(W, S). Coxeter system

W_J parabolic subgroup

There is a nice way to label the left cosets, because each has a unique elt of min length.

Def Parabolic quotient

$$W^J = \{w \in W : ws > w \text{ for } s \in J\}$$

$$= \{w \in W : \text{no reduced word ends in } s \in J\}$$

Prop. Every $w \in W$ factors uniquely as

$$w = w^J w_J \qquad w^J \in W^J,\ w_J \in W_J$$

and $\ell(w) = \ell(w^J) + \ell(w_J)$

(w_J^{-1} = things in J; w^J = (not shorten-able by J))

Pf **Existence.** Need: w (things in J) = (not shorten-able by J)

Shorten w successively by J until you can't:

$$w > w_{j_1} > w_{j_1 j_2} > \cdots > \underbrace{w_{j_1 \cdots j_k}}_{w^J}\ \underbrace{}_{w_J^{-1}} \qquad j_i \in J$$

Note $\ell(w) = \ell(w^J) + \ell(w_J)$

Uniqueness Sup. $w = w^J w_J = u^J u_J$

$$w^J = \underbrace{u^J u_J w_J^{-1}}$$

delete to a reduced word for w^J

can't end in things in J, so I can't use $u_J w_J^{-1}$

$\Rightarrow w^J \le u^J$ in Bruhat (subword prop)

Similarly $u^J \le w^J$ ∎

Corollary (W^J labels cosets of W_J)
The coset wW_J has a unique elt of min length, w^J.

Pf. ∘ $w^J = w w_J^{-1} \in W_J$ has length $\ell(w^J)$

∘ Any other $wv = w^J w_J v$ has $\ell(w^J) + \ell(w_J v)$ ∎
(v in W_J)

Corollary $W^J \times W_J$ is a subset of W
(same elts, fewer relations)

Ex S_3: a — b $\quad J = \{a\} = \{(12)\}$

$W_J = \{e, a\}$

$W^J = \{e, b, ab\}$