Corollary $u < v \iff uw_0 > vw_0$

Pf $u \to tu$ in Bruhat graph $\iff \ell(tu) > \ell(u)$
$\Updownarrow$
$uw_0 \leftarrow tuw_0$ in Bruhat graph $\iff \ell(tuw_0) < \ell(uw_0)$ ☒

So: $u \mapsto uw_0$ is a poset antiautomorphism
$u \mapsto w_0 u$ is a poset antiautomorphism
$u \mapsto w_0 u w_0$ is a poset automorphism.

Some questions

- For S_n, what does $u \mapsto uw_0$ do?
  $u \mapsto w_0 u$ do?
  $u \mapsto w_0 u w_0$ do?
- Is w_0 a reflection? $(w_0^2 = e)$

Parabolic Subgroups and Quotients

General subgroups of W are arbitrarily complicated, since any group is a subgroup of a sym. gp.

Q. Are infinite sym gps Cox gps?

So we focus on parabolic subgroups, which behave well.

(W, S) Coxeter system

For $J \subseteq S$ let W_J be the subgp gen by J.
These are the parabolic subgroups ($2^{|S|}$ of them.)

Properties

(i) (W_J, J) is a Coxeter system
(ii) $\ell_J(w) = \ell(w)$ for $w \in W_J$ ($\ell_J = \ell$ in (W_J, J))
(iii) $W_I \cap W_J = W_{I \cap J}$
(iv) $\langle W_I \cup W_J \rangle = W_{I \cup J}$
(v) $W_I = W_J \iff I = J$

Pf
(ii) Take a J-word for w, and reduce by deletion.
(i) By (ii), (W_J, J) inherits exchange from (W, S)
(iii)-(v): Exercises. ☒