The "long word"

Lemma. For every $u,v\in W$ there is $w\in W$ with $u\le w, v\le w$

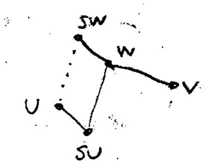
Pf. Induct on $\ell(u)+\ell(v)$.

- case 0: ✓
- case $n-1 \to$ case n:

  Take s with $\ell(su)=\ell(u)-1$.

  Take $su\le w$, $v\le w$

  If $w<sw$:
  lift to $u\le sw$
  $v\le sw$



  If $w>sw$:
  lift to $u\le w$
  $v\le w$ ☒

Corollary A finite Coxeter group has a unique max elt w_0, the "long word".

Converse: If there is $w_0\in W$ with $sw_0<w_0$ for all $s\in S$, then W is finite, w_0 is its max

Pf: $u\le w_0$ follows by induction on $\ell(u)$.
W is finite since all of W can be realized as subwords of a reduced word for w_0 ☒

For $W=S_n$, $w_0 = n\ n{-}1 \cdots 3\ 2\ 1$.

Recall:
$T_L(v)=\{\text{reflections } t \mid tv<v\}$
$\ell(v)=|T_L(v)|$

Properties.

(i) $w_0^2=e$

(ii) $\ell(ww_0)=\ell(w_0)-\ell(w)$

(iii) $T_L(ww_0)=T\setminus T_L(w)$

(iv) $\ell(w_0)=|T|$

Pf

(i) Since $\ell(w_0^{-1})=\ell(w_0)$, $w_0^{-1}\le w_0 \Rightarrow w_0^{-1}=w_0$

(ii) $\ge$: by triangle inequality

$\le$: Induct down on $\ell(w)$.

$\ell(w)=\ell(w_0)$: $w=w_0$ ✓

$\ell(w)=n\to n-1$:

Take $sw>w$

$$\ell(sww_0)\le\ell(w_0)-\ell(sw)$$
$$\ell(ww_0)-1 \le \ell(w_0)-\ell(w)-1 \quad ✓$$

(iii) $t\in T_L(ww_0) \iff tww_0<ww_0$
$\iff \ell(tww_0)<\ell(ww_0)$
$\iff \ell(tw)>\ell(w)$
$\iff tw>w$
$\iff t\notin T_L(w)$

(iv) Apply (iii) to $w=e$ ☒