Properties of Bruhat order

Recall

exchange: If $w = s_1 \cdots s_k$ and $\ell(wt) < \ell(w)$ in T

then $wt = s_1 \cdots \hat{s}_i \cdots s_k$ some i

$t = s_k \cdots s_i \cdots s_k$

right mult by t = eliminate s_i

deletion: If $w = s_1 \cdots s_k$ and $\ell(w) < k$ then

$w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$ some i, j

Theorem (Subword Property)
Let $w = s_1 \cdots s_q$ be reduced
$u \le w \iff u = s_{i_1} \cdots s_{i_k}$ is reduced
(some $1 \le i_1 < \cdots < i_k \le q$)

$\Rightarrow$ First sup $u \to w$

$w = ut$, $\ell(w) > \ell(u) = \ell(wt)$

$\Rightarrow u = wt = s_1 \cdots \hat{s}_i \cdots s_q$ subword

Now if $u \le w$

$u = u_0 \to u_1 \to \cdots \to u_s = w$

and each u_i is a subword of u_{i+1}

$\Rightarrow u$ is a subword of w ■

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$\Leftarrow$: Induct on $\ell(w) - \ell(u)$.

$\ell(w) - \ell(u) = 1$: $w = s_1 \cdots s_q$

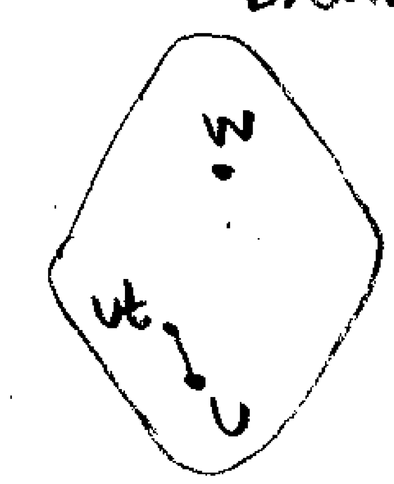
$u = s_1 \cdots \hat{s}_i \cdots s_q \Rightarrow u = w \underbrace{s_q \cdots s_i \cdots s_q}_{\text{in } T} \Rightarrow u \to w$

$\ell(w) - \ell(u) = k$: $w = s_1 \cdots s_q$

$u = s_1 \cdots \hat{s}_{a_1} \cdots \hat{s}_{a_2} \cdots \hat{s}_{a_k} \cdots s_q$

(Choose so a_k is max.)

Bruhat



$u \underbrace{s_q \cdots s_{a_k} \cdots s_q}_{t \text{ in } T} = s_1 \cdots \hat{s}_{a_1} \cdots \hat{s}_{a_{k-1}} \cdots s_q$

□ If $\ell(ut) = \ell(u) + 1$

- u reduced subword of $ut \Rightarrow u \le ut$
  $\ell(ut) - \ell(u) = 1$
- ut reduced subword of $u \Rightarrow ut \le w$
  $\ell(w) - \ell(ut) = 1$

$\Rightarrow u \le w$

□ If $\ell(ut) = \ell(u) - 1$

(a) $t = s_q \cdots s_i \cdots s_q$ for $i > a_k$

or (b) $\cancel{t} = s_q \cdots \hat{s}_{a_k} \cdots s_r \cdots \hat{s}_{a_k} \cdots s_q$

a) $w = w t t$

$= w(s_q \cdots s_{a_k} \cdots s_q)(s_q \cdots s_i \cdots s_q)$

$= s_1 \cdots \hat{s}_{a_k} \cdots \hat{s}_i \cdots s_q$

but $\ell(w) = q$!

b) $u = vtt$

$= (s_1 \cdots \hat{s}_{a_1} \cdots \hat{s}_{a_k} \cdots s_q)(s_q \cdots \hat{s}_{a_k} \cdots s_r \cdots \hat{s}_{a_k} \cdots s_q)(s_q \cdots s_{a_k} \cdots s_q)$

$= (s_1 \cdots \hat{s}_{a_1} \cdots \hat{s}_r \cdots \hat{s}_{a_k} \cdots s_q)(s_q \cdots s_{a_k} \cdots s_q)$

$= s_1 \cdots \hat{s}_{a_1} \cdots \hat{s}_r \cdots s_{a_k} \cdots s_q$

another reduced subword with $r < a_k \Rightarrow\Leftarrow$ ∎

Corollary

$u \le v \iff u^{-1} \le v^{-1}$ (by subword description)

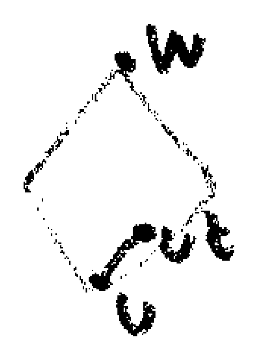
Corollary (Chain Property)

If $u < w$ then there are

$$u = v_0 < v_1 < v_2 < \cdots < v_k = w$$

lengths: $\ell(u)$, $\ell(u)+1$, $\ell(u)+2$, …, $\ell(u)+k = \ell(w)$

Pf By proof of above and ind. chain:



In other words,

Prop The Bruhat order is graded by length

This means it has "floors", not like 

Def: A poset is graded if every maxl chain from u to v ($u<v$) has same length

Def $u \lessdot v$ if $u<v$, and there is no w with $u<w<v$

v "covers" u - these are the segments we draw when we draw the Hasse diagram of a poset.

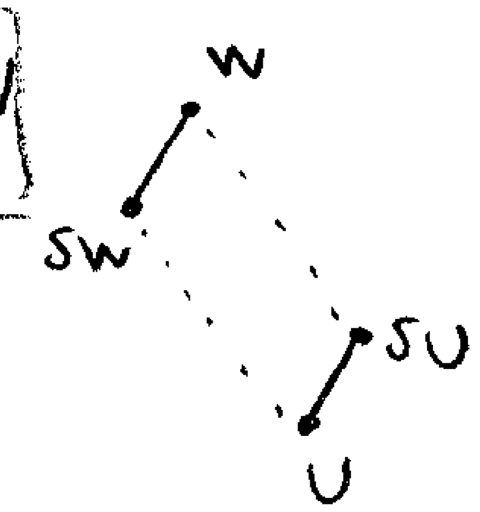
Pf of Prop:

By chain property, $u \lessdot v \Rightarrow \ell(v) = \ell(u)+1$ ∎

Prop. (Lifting Property) $u, w \in W$, $s \in S$

If $u < w$, $sw < w$, $u < su$

then $u \le sw$, $su \le w$



Take $sw = s_1 \cdots s_q$ reduced

$w = s s_1 \cdots s_q$ reduced

$u = s_{i_1} \cdots s_{i_k}$ subword of $s s_1 \cdots s_q$

Since $su > u$, $s \neq s_{i_1}$ so

$u = s_{i_1} \cdots s_{i_k}$ subword of $s_1 \cdots s_q \Rightarrow u \le sw$

Similarly $su \le w$ ∎