Le yeeOn $\frac{11}{16}$

#9. Let (W,S) be a Coxeter system.

Let $w = s_1 \cdots s_r \in W$ be a reduced word.

Prove that the +ve roots w sends to the −ve set are precisely the r roots $\beta_i = s_r s_{r-1} \cdots s_{i+1}(\alpha_i)$, i.e.

$$\beta_1 = s_r s_{r-1} \cdots s_2(\alpha_1)$$
$$\beta_2 = s_r s_{r-1} \cdots s_3(\alpha_2)$$
$$\vdots$$
$$\beta_{r-1} = s_r(\alpha_{r-1})$$
$$\beta_r = \alpha_r.$$

Pf:

$w = s_1 \cdots s_r \in W$ is reduced $\Rightarrow \ell(w) = r$.

By prop'n 4.4.4 (B & B), proven in class, the # of +ve roots assoc'd w/ w is exactly r.

So, we need only to show that $\begin{cases} \beta_i > 0 \\ w(\beta_i) < 0, \ i = 1, \dots, r \end{cases}$

β_i as specified above and that they're all distinct.

Note that $w(\beta_i) = (s_1 \cdots s_r) s_r s_{r-1} \cdots s_{i+1}(\alpha_i)$
$= s_1 \cdots s_i(\alpha_i)$

Since $s_1 \cdots s_r$ is reduced, all $s_1 \cdots s_k$ (prefixes) are also reduced. $k \leq r$

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Consequently,

$$\ell((s_1 \cdots s_i) s_i) = \ell(s_1 \cdots s_{i-1}) = i-1 < i = \ell(s_1 \cdots s_i)$$

and

$$\ell((s_r \cdots s_{i+1}) s_i) = r-i+1 > r-i = \ell(s_r \cdots s_{i+1}).$$

Thus, by prop'n 4.2.5 (ii) B & B,

$$w(\beta_i) = (s_1 \cdots s_i)\alpha_i < 0$$

$$\beta_i = (s_r \cdots s_{i+1})\alpha_i > 0, \quad i = 1, \cdots, r.$$

Now we prove β_i are all distinct.

Assume not: Then let $j > i$ but $\beta_j = \beta_i$.

Then $s_r \cdots s_{j+1}(\alpha_j) = s_r \cdots s_{i+1}(\alpha_i)$

Which implies $\alpha_j = s_j \cdots s_{i+1}(\alpha_i)$.

So, $-\alpha_j = s_j \alpha_j = s_{j-1} \cdots s_{i+1}(\alpha_i)$. (*)

In the case $j = i+1$, $-\alpha_j = \alpha_i$. (**)

In all cases, since $s_{j-1} \cdots s_{i+1} s_i \geq s_{j-1} \cdots s_{i+1}$ (in length),

the RHS of (*) is > 0 but the LHS of (*) is < 0

since all simple roots are > 0, which $\Rightarrow$ (**) is $\Rightarrow\Leftarrow$

$\therefore$ $(\Rightarrow\Leftarrow)$ in general. QED.