(7) Given a cube of side length $\sqrt{2}$, show that the 12 vectors from the center of cube to the midpoints of its 12 edges form a root system of S_4. Then draw some pictures.

Proof: Let a, b, c denote $s_1, s_2, s_3 \in S_4$, respectively and let +, - denote $\frac{1}{\sqrt{2}}$, $-\frac{1}{\sqrt{2}}$, respectively. We associate a with the vector $\alpha_a = (0, -, -)$, b with $\alpha_b = (+, 0, +)$, and c with $\alpha_c = (0, +, -)$. First we need to check that $\langle \alpha_a, \alpha_b \rangle = \langle \alpha_b, \alpha_c \rangle = -\cos\frac{\pi}{3} = -\frac{1}{2}$

and that $\langle \alpha_a, \alpha_c = -\cos\frac{\pi}{2} = 0$.

Since S_4 is finite, our inner product is the standard inner product. Thus if θ is the angle between two vectors u and v, then $\cos\theta = \frac{\langle u,v \rangle}{|u||v|}$

So, $\langle \alpha_a, \alpha_b \rangle = 0 + 0 - \frac{1}{2} = -\frac{1}{2} = \cos(\theta_1)$ and

$\langle \alpha_b, \alpha_c \rangle = 0 + 0 - \frac{1}{2} = -\frac{1}{2} = \cos(\theta_2)$ and

$\langle \alpha_a, \alpha_c \rangle = 0 - \frac{1}{2} + \frac{1}{2} = -0 = \cos(\theta_3)$.

Thus $\theta_1 = \theta_2 = \frac{\pi}{3}$ and $\theta_3 = \frac{\pi}{2}$ as desired.

So $\alpha_a, \alpha_b, \alpha_c$ are the simple roots for the root system of S_4.

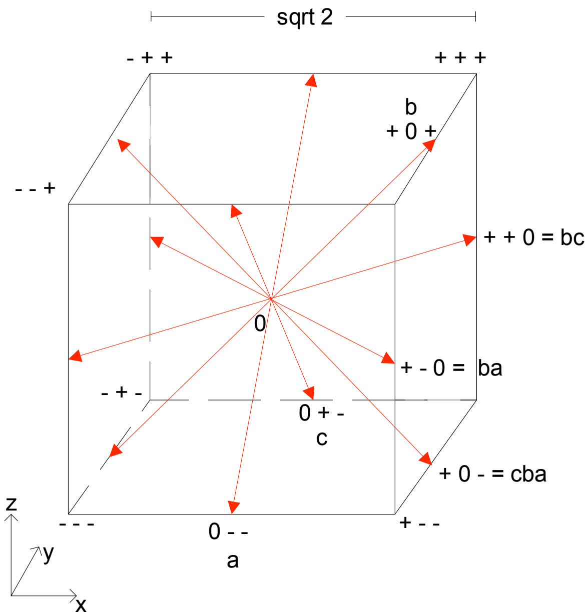
Finally, we show that these three roots generate the other edges.

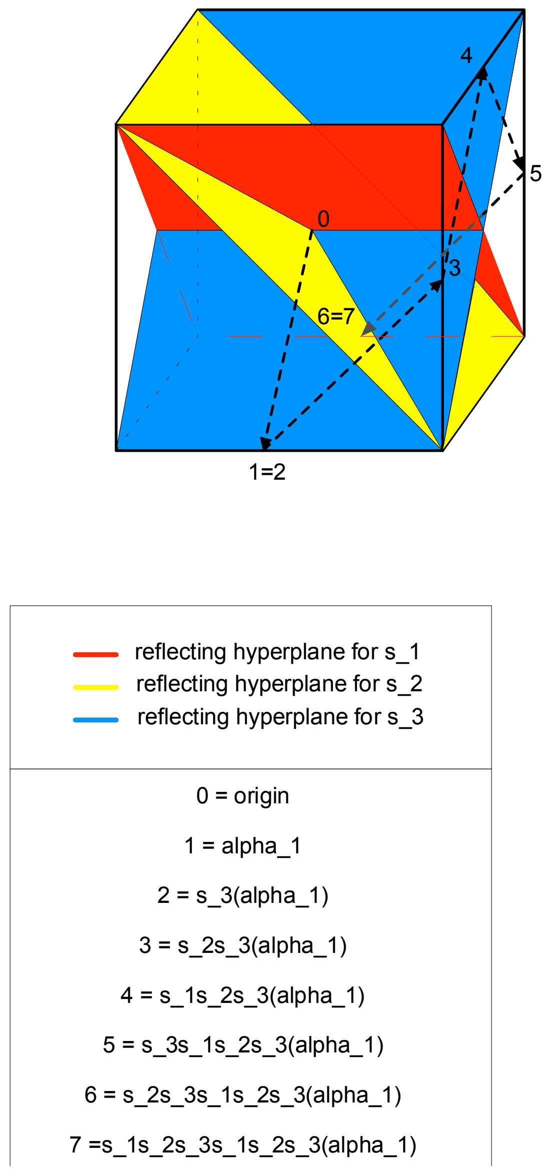
$\alpha_a + \alpha_b = (+, -, 0)$,

$\alpha_b + \alpha_c = (+, +, 0)$, and

$\alpha_a + \alpha_b + \alpha_c = (+, 0, -)$.

These are exactly the six positive roots (see figure below) and the other six edges are obtained by multiplying each by -1.


sqrt 2
- + +
+ + +
b
+ 0 +
- - +
+ + 0 = bc
0
+ - 0 = ba
- + -
0 + -
c
+ 0 - = cba
z
y
x
- - -
0 - -
a
+ - -


4
5
0
3
6=7
1=2
reflecting hyperplane for s_1
reflecting hyperplane for s_2
reflecting hyperplane for s_3
0 = origin
1 = alpha_1
2 = s_3(alpha_1)
3 = s_2s_3(alpha_1)
4 = s_1s_2s_3(alpha_1)
5 = s_3s_1s_2s_3(alpha_1)
6 = s_2s_3s_1s_2s_3(alpha_1)
7 =s_1s_2s_3s_1s_2s_3(alpha_1)