4 We'll look at the left weak order. This choice is of no substance; $x \leq y$ in the left weak order iff $x^{-1} \leq y^{-1}$ in the right weak order.

In the left weak order on the symmetric group S_n, given any $w \in S_n$ and any s_i, there exists a covering relation in one direction or the other between w and $s_i w$, according to whether $l(s_i w)$ is greater or less than $l(w)$. In either case, the graph arising from the covering relations has an edge $(w, s_i w)$. These are all the covering relations, so these are all the edges.

Left-multiplication by a Coxeter generator $s_i = (i\ i+1)$ has the effect of switching which elements are mapped to i and $i+1$, i.e. exchanging two consecutive integers i and $i+1$ wherever they appear in the permutation when written out as a list of images. This then is the condition for two permutations $\pi, \pi' \in S_n$ to be joined by an edge in the weak order.

We claim the same condition identifies the edges of the permutahedron Π_n. We'll identify a permutation π with the point $(\pi(1), \ldots, \pi(n))$ whose coordinates it gives.

The edges of Π_n are by definition exactly the one-dimensional faces, i.e. the one-dimensional subpolytopes which can be realised as the intersection of Π_n with a hyperplane. So let be given two permutations $\pi, \pi' \in S_n$, and let H be a hyperplane that intersects Π_n only in the edge (π, π'), i.e. such that one of the two open halfspaces delimited by H contains all permutations other than π and π'.

Choosing a normal vector a to H of suitable sign, we may assume $\langle \pi, a \rangle = \langle \pi', a \rangle =: b$ and $\langle \sigma, a \rangle < b$ for any other permutation $\sigma \in S_n$. Thus for any $i < j$ in $[n]$ we must have $a_{\pi^{-1}(i)} \leq a_{\pi^{-1}(j)}$, otherwise the permutation $(i\ j)\pi$, which switches the preimages of i and j, would achieve $\langle (i\ j)\pi, a \rangle > \langle \pi, a \rangle$. For the same reason we have strict inequality $a_{\pi^{-1}(i)} < a_{\pi^{-1}(j)}$ unless $\pi' = (i\ j)\pi$. The same is true using π' in place of π. Therefore at least two entries of a must be equal, since there's at least one pair $i < j$ such that $\pi^{-1}(i)$ and $\pi^{-1}(j)$ compare in the opposite sense to $\pi'^{-1}(i)$ and $\pi'^{-1}(j)$. Thus we do have an equality of form $\pi' = (i\ j)\pi$. Furthermore, i and j must be consecutive integers, otherwise we'd have $a_{\pi^{-1}(i)} < a_{\pi^{-1}(i+1)} < a_{\pi^{-1}(j)} = a_{\pi^{-1}(i)}$.

This shows that any edge (π, π') of Π_n has $\pi' = (i\ i{+}1)\pi$ for some i. This is in fact also a sufficient condition. We claim that if we choose a to be any vector with its entries suitably ordered, i.e. such that $a_{\pi^{-1}(i)} = a_{\pi^{-1}(i+1)}$ and $a_{\pi^{-1}(j)} < a_{\pi^{-1}(j+1)}$ for $j \neq i$, and take b to be $\langle \pi, a \rangle = \langle \pi', a \rangle$, we will realise the edge (π, π') as the face defined by the hyperplane $\langle \sigma, a \rangle = b$.

For any $\sigma_0 \in S_n$ distinct from π, π', there will be some pair $i' < j'$ in $[n]$ not equal to $(i, i+1)$ such that $\sigma_0(\pi^{-1}(i)) > \sigma_0(\pi^{-1}(i+1))$; and then $\sigma_1 := (i'\ j')\sigma_0$ satisfies $\langle \sigma_1, a \rangle > \langle \sigma_0, a \rangle$. Iterating in this way, define a sequence of permutations (σ_k): if ever some σ_k is π or π', let the sequence terminate; otherwise construct σ_{k+1} so that $\langle \sigma_{k+1}, a \rangle > \langle \sigma_k, a \rangle$. If ever the sequence does terminate, by transitivity among all these inequalities we get $\langle \sigma_0, a \rangle < b$. Otherwise, since S_n is finite we must have distinct indices k, k' with $\sigma_k = \sigma_{k'}$; but then transitivity of inequalities gives the contradiction $\langle \sigma_{k'}, a \rangle > \langle \sigma_k, a \rangle$. We conclude that every permutation σ other than π and π' lies in the open halfspace $\langle \sigma, a \rangle < b$, as claimed.