Exercise 2 Let $(S_n)^J$ be a parabolic quotient of S_n modulo a set of generators J with $|J| = n-2$. Prove that, if $x, y \in (S_n)^J$, then

$$x \leq y \text{ in the Bruhat order } \iff x \leq_L y \text{ in the left weak order.}$$

Proof. First, let $x \leq_L y$. Then x is the suffix of some reduced expression for y, i.e. $x = s_1 s_2 \ldots s_r$ and $y = s'_1 s'_2 \ldots s'_t s_1 s_2 \ldots s_r$. Hence, x is a subword of y and thus $x \leq y$ in the Bruhat order.

Now, suppose $x \leq y$ in the Bruhat order. By assumption, $x, y \in (S_n)^J$. By Theorem 2.5.5 we know $(S_n)^J$ is graded; in other words there exist elements $w_i \in (S_n)^J$, $l(w_i) = l(x) + i$ for $0 \leq i \leq k$ such that $x = w_0 \leq w_1 \leq \cdots \leq w_k = y$. To show $x \leq_L y$, we will do induction on $l(y) - l(x)$. If $l(y) - l(x) = 0$ then $x = y$ and so $x \leq_L y$. Assume that if $l(y) - l(x) = k$ we know $x \leq_L y$ and now consider $l(y) - l(x) = k + 1$. By Theorem 2.5.5 there exists a $u \in (S_n)^J$ such that, $x < u < y$ and $l(u) = l(x) + 1$, so $l(y) - l(u) = k$, thus $u \leq_L y$ by out inductive hypothesis. By Lemma 2.1.4, u covers $x = x_1 x_2 x_3 \ldots x_i \ldots x_j \ldots x_n$ in the Bruhat order iff $u = x_1 x_2 \ldots x_j \ldots x_i \ldots x_n$ such that $x_i < x_j$. In $(S_n)^J$ it is only possible to switch consecuvtive values, because of the "wall" property seen in Homework 3, Exercise 2. Thus $j = i + 1$ which corresponds to a simple reflection being multiplied on the left of x to get u. Therefore, $x \leq_L u$ and so $x \leq_L y$.