Exercise 6 Given a Coxeter system (W, S) and $I, J \subseteq S$, prove that

(a) $W_I \cap W_J = W_{I \cap J}$
(b) $\langle W_I \cup W_J \rangle = W_{I \cup J}$
(c) $W_I = W_J$ if and only if $I = J$.

Proof. .

(a) Pick $x \in W_I \cap W_J$. Then $x \in W_I$ and $x \in W_J$, so $x = s_1 \dots s_m$ where $s_1, \dots, s_m \in I$ and $x = r_1 \dots r_n$ where $r_1, \dots, r_n \in J$. Then by Corollary 1.4.8(ii), $\{r_1, \dots r_n\} = \{s_1, \dots s_m\}$, therefore, $\{r_1, \dots r_n\} \in I \cap J$ and $x \in W_{I \cap J}$, hence $W_I \cap W_J \subseteq W_{I \cap J}$.
Next, pick some $x \in W_{I \cap J}$. Then $x = s_1 \dots s_n$ where $s_i \in I \cap J$. Hence, $x \in W_I$ and $x \in W_J$, therefore $x \in W_I \cap W_J$. So $W_{I \cap J} \subseteq W_I \cap W_J$, thus $W_I \cap W_J = W_{I \cap J}$.

(b) Pick $x \in \langle W_I \cup W_J \rangle$. Then, x is in the group generated by the union of the parabolic subgroups W_I and W_J. So $x = s_1 s_2 \dots s_k$ such that either $s_i \in I$ or $s_i \in J$ for all $i \in \{1, \dots, k\}$, because x can be any combination of elements from W_I and W_J. Then for each s_i it follows that $s_i \in I \cup J$, therefore $x \in W_{I \cup J}$. Hence $\langle W_I \cup W_J \rangle \subseteq W_{I \cup J}$.
Now, choose some $x \in W_{I \cup J}$. Then $x = s_1 s_2 \dots s_k$ such that $s_i \in I \cup J$ for all $i \in \{1, \dots, k\}$. Therefore either $s_i \in I$ or $s_i \in J$. By above, all elements of $\langle W_I \cup W_J \rangle$ are made up of generators such that either $s_i \in I$ or $s_i \in J$, thus $x \in \langle W_I \cup W_J \rangle$ and $W_{I \cup J} \subseteq \langle W_I \cup W_J \rangle$.

(c) Suppose $W_I = W_J$ but $I \neq J$. Then there must exist some $s_i \in I$ such that $s_i \notin J$. Then by definition $w = s_i \in W_I$ and by our assumption $w \in W_J$. Therefore $w = s'_1 \dots s'_r$ for some $s_j \in I$. However, this implies $s_i = s'_1 \dots s'_r$ which is a contradiction because by Corollary 1.4.8(iii) no Coxeter generator can be expressed in terms of others. Therefore if $W_I = W_J$ then $I = J$. The opposite direction follows from the definition of W_I given a set I; if $I = J$ then $W_I = W_J$.
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