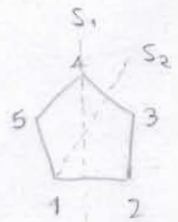
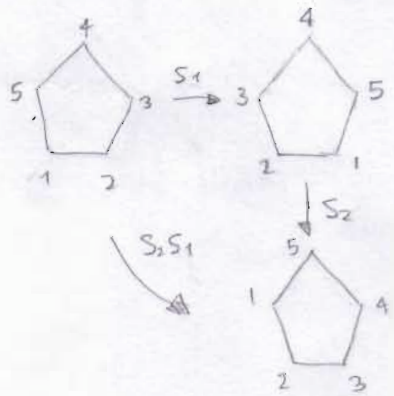
4. Prove that every dihedral group is a Coxeter group

Solution

We prove that D_n (the dihedral group of order $2n$) has the representation $\langle S_1, S_2 \mid S_1^2 = S_2^2 = (S_1S_2)^n = 1 \rangle$ as a Coxeter Group.



We identify the generators S_1 and S_2 with 'adjacent' reflections, (as shown in the graphic for $n=5$) then the composition S_2S_1 rotates clockwise the regular polygon $\frac{360^\circ}{n}$ degrees.



It is clear that we can generate any other of the remaining symmetries of the polygon, since these consist of reflections and rotations and we already have the basic 'blocks': to generate a reflection through a line that passes through other vertex, we just rotate the polygon accordingly, and reflect.

Hence, the $2n$ symmetries can be generated as 'words' in S_1, S_2. We now prove that under the relations $S_1^2, S_2^2, (S_1S_2)^n = 1$ no more than $2n$ nonequivalent words can be written.

Starting with S_1, we can write $S_1, S_1S_2, S_1S_2S_1, \dots, \underbrace{S_1S_2\dots S_k}_{n-1 \text{ letters}}$, $n-1$ different words. We alternate letters since $(S_i)^2 = e$. Similarly, starting with S_2 there are $n-1$ different words. Counting the empty word (the identity) and substracting one since $\underbrace{S_1S_2\dots}_{n-1 \text{ letters}} = \underbrace{S_2S_1\dots}_{n-1 \text{ letters}}$, we have $2n$ words.

Longer words can be shortened: If a word doesn't have two adjacent repeated letters, then it must have an alternating sequence of length n, which is the identity. $\blacksquare$