3. (a) The equivalence relation $\approx$ we really want to work with is the transitive closure of the relation $\sim$ described in this problem statement (which is not transitive).

To show that $S^*/\approx$ inherits a product from S^* it is enough to show that if $x, x', y, y' \in S^*$ are such that $x \sim x'$ and $y \sim y'$ then $xy \approx x'y'$ (this easily implies that the product is well defined on the equivalence classes of $\approx$). But if $x \sim x'$ then $x = uv$ and $x' = u(s_1s_2)^{m(s_1,s_2)}v$ (or vice versa) for some $u, v \in S^*$ and $s_1, s_2 \in S$, and also if $y \sim y'$ then $y = pq$ and $y' = p(s_3s_4)^{m(s_3,s_4)}q$ (or vice versa) for some $p, q \in S^*$ and $s_3, s_4 \in S$. Therefore $xy \approx x'y'$, since all the possible cases are

a) $xy = uvpq \approx u(s_1s_2)^{m(s_1,s_2)}vp(s_3s_4)^{m(s_3,s_4)}q = x'y'$,

b) $xy = uvp(s_3s_4)^{m(s_3,s_4)}q \approx u(s_1s_2)^{m(s_1,s_2)}vpq = x'y'$,

c) $xy = u(s_1s_2)^{m(s_1,s_2)}vpq \approx uvp(s_3s_4)^{m(s_3,s_4)}q = x'y'$ or

d) $xy = u(s_1s_2)^{m(s_1,s_2)}vp(s_3s_4)^{m(s_3,s_4)}q \approx uvpq = x'y'$.

equivalence class of the word $s_1s_2\dots s_{k-1}s_k$ is the equivalence class of the word $s_ks_{k-1}\dots s_2s_1$, since $s_1s_2\dots s_{k-1}s_ks_ks_{k-1}\dots s_2s_1 \approx \emptyset$). Therefore $S^*/\approx$ is a group.

Now, consider the function $f' : S \to S^*/\approx$ defined as $f'(s) = \overline{s}$ (where $\overline{s}$ is the equivalence class of s). By the universal property of free groups this function can be extended to a group homomorphism $f : F \to S^*/\approx$, which is clearly surjective. We will show that $\ker(f) = N$, so $W \cong F/N \cong S^*/\approx$ as we want.

- $\ker(f) \supseteq N$: Note that for every $s, s' \in S$ we have that $f((ss')^{m(s,s')}) = \overline{(ss')^{m(s,s')}} = \overline{\emptyset}$, so $\{(ss')^{m(s,s')} : s, s' \in S\} \subseteq \ker(f)$. Then, since $\ker(f)$ is a normal subgroup of F, we get that $N \subseteq \ker(f)$.
- $\ker(f) \subseteq N$: Assume $x = s_1^{\pm 1} s_2^{\pm 1} \dots s_k^{\pm 1} \in F$ is in $\ker(f)$. This means that

$$\begin{aligned}\overline{\emptyset} &= f(x)\\ &= f(s_1^{\pm 1} s_2^{\pm 1} \dots s_k^{\pm 1})\\ &= \overline{s_1}^{\pm 1}\,\overline{s_2}^{\pm 1} \dots \overline{s_k}^{\pm 1}\\ &= \overline{s_1}\,\overline{s_2} \dots \overline{s_k}\\ &= \overline{s_1 s_2 \dots s_k},\end{aligned}$$

  so $s_1s_2\dots s_k \approx \emptyset$. The result follows from the following facts:

  - If $x, y \in S^*$ are such that $x \sim y$ and $x \in N$ (regarded as an element of F), then $y \in N$:
    If $x \sim y$ then $x = uv$ and $y = u(ss')^{m(s,s')}v$ (or vice versa) for some $u, v \in S^*$ and $s, s' \in S$. Then

$$\begin{aligned}y &= u(ss')^{m(s,s')}v\\ &= u(vv^{-1})(ss')^{m(s,s')}v\\ &= (uv)(v^{-1}(ss')^{m(s,s')}v)\\ &= x(v^{-1}(ss')^{m(s,s')}v) \in N\end{aligned}$$

    or

$$\begin{aligned}y &= uv\\ &= u((ss')^{m(s,s')}vv^{-1}(ss')^{-m(s,s')})v\\ &= (u(ss')^{m(s,s')}v)(v^{-1}(ss')^{-m(s,s')}v)\\ &= x(v^{-1}(ss')^{-m(s,s')}v) \in N.\end{aligned}$$

  - If $x = s_1s_2\dots s_k \in N$ then $y = s_1^{\pm 1} s_2^{\pm 1} \dots s_k^{\pm 1} \in N$:
    Note that

$$y = (s_1s_2\dots s_{k-1}s_k\, s_k^{-1}s_{k-1}^{-1}\dots s_2^{-1}s_1^{-1})\, y = x(s_k^{-1}s_{k-1}^{-1}\dots s_2^{-1}s_1^{-1}\, y).$$

    We will show by induction on k that for any $y = s_1^{\pm 1} s_2^{\pm 1} \dots s_k^{\pm 1} \in F$ we have that $z = s_k^{-1}s_{k-1}^{-1}\dots s_2^{-1}s_1^{-1}\, y \in N$, which clearly implies what we want. If $k = 1$ the result is clear. Now, assume the result holds for $k-1$. Since we can cancel terms in the middle until we get

$$z = s_k^{-1} s_{k-1}^{-1} \dots s_{l+1}^{-1} \, s_l^{-2} \, s_{l+1}^{\pm 1} \dots s_{k-1}^{\pm 1} s_k^{\pm 1}$$

for some $l \geq 1$. But then

$$\begin{aligned} z &= s_k^{-1} s_{k-1}^{-1} \dots s_{l+1}^{-1} \, s_l^{-2} \, (s_{l+1} \dots s_{k-1} s_k \, s_k^{-1} s_{k-1}^{-1} \dots s_{l+1}^{-1}) \, s_{l+1}^{\pm 1} \dots s_{k-1}^{\pm 1} s_k^{\pm 1} \\ &= (s_k^{-1} s_{k-1}^{-1} \dots s_{l+1}^{-1} \, s_l^{-2} \, s_{l+1} \dots s_{k-1} s_k)(s_k^{-1} s_{k-1}^{-1} \dots s_{l+1}^{-1} \, s_{l+1}^{\pm 1} \dots s_{k-1}^{\pm 1} s_k^{\pm 1}), \end{aligned}$$

which by induction hypothesis is a product of two elements of N, and we are done.

3. (b) Assume G is a group and $f : S \to G$ is a function such that $(f(s)f(s'))^{m(s,s')} = e$ for all $s, s' \in S$. By the universal property of free groups, f extends uniquely to a group homomorphism $f' : F \to G$. Now, our assumption about f clearly implies that $\ker(f') \supseteq N$, so f' factors through a (unique) group homomorphism $f'' : F/N \cong W \to G$, and the proof is complete.