Problem 8. Let $X_{2n} = \{x \in S_{2n} : |x(i) - i| \leqslant n \text{ for } 1 \leqslant i \leqslant 2n\}$. We are describing tables in matrix coordinates so, for example, if we have the dots table for ϕ and a dot in position ij, that means $\phi(i) = 2n + 1 - j$.

Result 1. Let (χ_{ij}) be a dots table for $x \in S_{2n}$. Then, the following conditions are equivalent:

i. $x \in X_{2n}$
ii. If χ_{ij} contains a dot, then $n < i + j \leqslant 3n + 1$

i $\Rightarrow$ ii. Let χ_{ij} be a cell with a dot, then $|x(i) - i| = |2n + 1 - (j + i)| \leqslant n$ so $-n \leqslant 2n + 1 - (i + j) \leqslant n$, but the left side is just $i + j \leqslant 3n + 1$ and the right one is $n + 1 \leqslant i + j$.
ii $\Rightarrow$ i. It is the same prove backwards.
Now, we claim that $X_{2n} = [id_{S_{2n}}, \phi]$ where $\phi(i) = n+i$ for $i \leqslant n$ and $\phi(i) = i-n$ for $i > n$. Let $x \in X_{2n}$, (x_{ij}) be the rank table for x and (χ_{ij}) the dots table. We try to find upper bounds for the value in each entry of (x_{ij}). First, we write,

$$(x_{ij}) = \begin{pmatrix} A_1 & M_1 \\ M_2 & A_2 \end{pmatrix} \text{ where } A_1, A_2, M_1, M_2 \text{ are } n \times n \text{ subtables}$$

- So let's work in A_1.
Take $\chi_{k_1k_2}$ with $k_1, k_2 \leqslant n$. If $k_1 + k_2 \leqslant n$, then it is easy to see that the North-West corner of $\chi_{k_1k_2}$ contains no dots. So, suppose $k_1 + k_2 > n$. We claim that $x_{k_1k_2} \leqslant k_1 + k_2 - n$. To prove it, suppose $x_{k_1k_2} > k_1 + k_2 - n$. Because (χ_{ij}) is a dots table for a permutation, we know there are $x_{k_1k_2}$ columns in $NWC(\chi_{k_1k_2})$ with dots, but the set $\{k_2, ..., n+1-k_1\}$ (Notice $k_2 \geqslant n+1-k_1$) has $k_1 + k_2 - n$ elements so the pigeonhole principle tells us that some entry χ_{ij} of $NWC(\chi_{k_1k_2})$ contains a dot with $i \leqslant k_1$, $j < n + 1 - k_1$ which implies $i + j < k_1 + n + 1 - k_1 = n + 1$ or $i + j \leqslant n$. But $x \in X_{2n}$ and that contradicts Result 1.

- Let's see M_1.
Take $\chi_{k_1k_2}$ with $k_1 \leqslant n, k_2 > n$. Then we have $k_1 < k_2$, so $x_{k_1k_2} \leqslant k_1$ is immediate.

- For M_2.

By symmetry with M_1, $x_{k_1k_2} \leqslant k_2$.

- We study A_2.
Take $\chi_{k_1k_2}$ with $k_1, k_2 > n$. This time we change a bit. Instead of looking for some upper bound for $x_{k_1k_2}$, we look for a lower bound for $n - x_{k_1k_2}$. Let

$$\chi_{k*} = \{(k,j) : \chi_{kj} \text{ contains a dot }\},\ \chi_{*k} = \{(i,k) : \chi_{ik} \text{ contains a dot }\}$$

We suppose $k_1, k_2 < 2n$ because by exercise 7, the numbers $x_{k_1(2n)}$ and $x_{(2n)k_2}$ are well known. Now, define the sets $A_{k_1} = \bigcup_{i=k_1+1}^{2n} \chi_{i*}$ and $B_{k_2} = \bigcup_{j=k_2+1}^{2n} \chi_{*j}$. Then we have,

$$2n - x_{k_1k_2} = \sharp(A_{k_1} \bigcup B_{k_2})$$

By the inclution-exclution principle,

$$2n - x_{k_1k_2} = \sharp A_{k_1} + \sharp B_{k_2} - \sharp(A_{k_1} \bigcap B_{k_2}) = (2n - k_1) + (2n - k_2) - \sharp(A_{k_1} \bigcap B_{k_2}).$$

So we need an upper bound for $\sharp(A_{k_1} \bigcap B_{k_2})$, but that is good news because one good bound is $\sharp\{i > k_1 : \text{there exists } j > k_2 \text{ satisfying } i + j \leqslant 3n + 1\}$. For $k_1 + k_2 + 1 \geqslant 3n + 1$ this number is just 0 and for $k_1 + k_2 + 1 < 3n + 1$, the number is $3n - (k_1 + k_2)$. So,

i. If $k_1 + k_2 \geqslant 3n$, $\sharp(A_{k_1} \bigcap B_{k_2}) = 0$
ii. If $k_1 + k_2 < 3n$, $\sharp(A_{k_1} \bigcap B_{k_2}) \leqslant 3n - (k_1 + k_2)$

Plugging this in our equation we get,

i. If $k_1 + k_2 \geqslant 3n$, $x_{k_1k_2} = k_1 + k_2 - 2n$
ii. If $k_1 + k_2 < 3n$, $x_{k_1k_2} \leqslant n$

So let's see what we have so far. If $x \in X_{2n}$ and (x_{ij}) is the rank table for x, then

1. $i, j \leqslant n$

   - If $i + j \leqslant n$, $x_{ij} = 0$
   - If $i + j > n$, $x_{ij} \leqslant i + j - n$

2. $i \leqslant n, j > n$

$$x_{ij} \leqslant i$$

3. $i > n, j \leqslant n$

$$x_{ij} \leqslant j$$

4. $i, j > n$

$$\text{- If } i+j \geqslant 3n,\ x_{ij} = i+j-2n$$
$$\text{- If } i+j < 3n,\ x_{ij} \leqslant n$$

But assuming the equalities in those equations, we get the entries of (ϕ_{ij}), the rank table for ϕ. That proves $X_{2n} \subseteq [id_{S_{2n}}, \phi]$. The other part is much shorter. Assume y does not belong to X_{2n}, let (y_{ij}) be the rank table for y and (γ_{ij}) the dots table. Then the cell γ_{ij} contains a dot with $i+j \leqslant n$ or $i+j > 3n+1$. If $i+j \leqslant n$, then $y_{ij} \geqslant 1$ but $\phi_{ij} = 0$, so either $y > \phi$ or they are not comparable. If $i+j > 3n+1$ with $i, j < 2n$, we build again our sets A_i, B_j and this time we obtain a lower bound for $\sharp(A_i \bigcap B_j)$. But that is immediate because a lower bound is 0 and then $2n = 0+2n \leqslant \sharp(A_i \bigcap B_j) + 2n = (2n-i)+(2n-j)+y_{ij}$ so $i+j-2n \leqslant y_{ij}$. But assuming $y \leqslant \phi$, we obtain $i+j-2n \geqslant y_{ij}$ so $i+j-2n = y_{ij}$. The problem here is that γ_{ij} contains a dot, which implies $y_{(i-1)(j-1)} = y_{ij} - 1$. We had $i+j > 3n+1$, so $(i-1)+(j-1) \geqslant 3n$ and we know that implies $\phi_{(i-1)(j-1)} = (i-1)+(j-1)-2n = y_{ij} - 2$ giving us $y_{(i-1)(j-1)} > \phi_{(i-1)(j-1)}$. The case for i or j equal to $2n$ is solved using the same argument and the fact that $s_{(2n)k} = s_{k(2n)} = k$ for all $s \in S_{2n}$. We proved $[id_{S_{2n}}, \phi] \subseteq X_{2n}$, so

$$[id_{S_{2n}}, \phi] = X_{2n}$$

Finally, we count atoms and coatoms. The number of atoms is easily seen to be $2n-1$ because there are $2n-1$ generators in the coxeter group S_{2n}, so there are exactly $2n-1$ covers of $id_{S_{2n}}$ in S_{2n} and all of them are smaller than ϕ in the Bruhat order. Now, let's write the complete representation of ϕ,

$$\phi = (n+1)(n+2)(n+3)...(2n)(1)(2)(3)...(n)$$

We group the numbers like,

$$\phi = \langle (n+1)(n+2)(n+3)...(2n)\rangle\langle (1)(2)(3)...(n)\rangle$$

We know the reflections of permutation groups are the transpositions, so let's se what happens if transpose the elements of our representation. If we transpose two elements of the first factor, the inversion number of ϕ will rise and then, the permutation obtained would be bigger or not comparable to ϕ. The same will happen if we transpose two elements of the second factor. Now if we choose any position a in the first factor and any position c in the second factor, there is no b such that $a < b < c$ and $\phi(a) > \phi(b) > \phi(c)$, so transposing positions a and c we decrease the inversion number of ϕ by one, so ϕ is a cover for all the permutations obtained in those cases. There are n numbers in the first factor and n umbers in the second one, so there are n^2 different permutations covered by ϕ, that is, n^2 coatoms.