#2

For a Coxeter system (W, S) let $l : W \to \mathbb{N}$ take $w \to k$ where $w = s_1 \dots s_k$ is a reduced expression for w. Then

(a) $l(uv) \equiv l(u) + l(v) \pmod 2$.

Proof. Let $u = s_1 \dots s_k$ and $v = s_{k+1} \dots s_n$ be reduced expressions for u and v. If $uv = s_1 s_2 \dots s_n$ is a reduced expression for uv, then $l(uv) = l(u) + l(v)$ and the result follows. On the other hand, if $uv = s_1 s_2 \dots s_n$ is not reduced, then we repeatedly apply the deletion property to obtain a reduced expression for uv. Each application of the deletion property preserves the parity of n (since it removes two letters at a time), so $l(uv) \equiv l(u) + l(v) \pmod 2$.

(b) $l(w^{-1}) = l(w)$

Proof. Since $l(ww^{-1}) = l(e) = 0$, we have that $l(w) \equiv l(w^{-1}) \pmod 2$, by part (a). Suppose $w = s_1 \dots s_k$ and $w^{-1} = t_1 \dots s_{k-2m}$ are reduced expressions for w and w^{1-} (where $m \in \mathbb{Z}_{\geq 0}$). We show that $m = 0$. Since $e = ww^{-1} = s_1 \dots s_k t_1 \dots t_{k-2m}$ has length zero, we apply deletion repeatedly to the RHS. But each application must remove an s_i and a t_j since our original expressions for w and w^{-1} were reduced. Therefore, to obtain the empty word we must apply deletion precisely k times. Thus, m=0 as desired.

(c) $l(sw) = l(w) \pm 1$

Proof. If $w = s_1 \dots s_k$ be a reduced expression for w. If sw is reduced, then clearly $l(sw) = l(w) + 1$. If sw is not reduced, then $sw = s_1 \dots \widehat{s_i} \dots s_k$ for some i by the exchange property. This is a reduced expression for sw since otherwise $w = s(sw)$ would be expressible as a word of length less than k which contradicts $w = s_1 \dots s_k$ being reduced.

(d) The distance function $d(u, v) = l(uv^{-1})$ is a metric on W.

Proof. Since no word can have negative length, the distance function is non-negative. Suppose $d(u, v) = 0$. Then $l(uv^{-1})$=0 implies that $u = (v^{-1})^{-1} = v$. Thus d is positive-definite. Since $vu^{-1} = (uv^{-1})^{-1}$, we have $l(uv^{-1}) = l(vu^{-1})$ by part (b), and hence that d is symmetric. Finally, if $s_1 \dots s_k$ and $t_1 \dots t_j$ are reduced expressions for uw^{-1} and wv^{-1} respectively, then we have a (not necessarily reduced) expression $uv = uw^{-1}wv^{-1} = s_1 \dots s_k t_1 \dots t_j$ of length $l(uw^{-1}) + l(wv^{-1})$. So $d(u, v) \leq d(u, w) + d(w, v)$. So d is a metric on W.