coxeter groups

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homework four

(due wednesday march 19 in sf, march 26 in bog.)

Instructions. Turn in your **five** best problems out of the following list. You are encouraged to work together on the homework, but you must state who you worked with on each problem. You *must* write your solutions independently and in your own words. I encourage you to type your solutions using LaTeX; that will be good practice for the final project, where LaTeX is mandatory.

1. (Thanks to Richard Stanley.) Let S_n be the symmetric group and w_0 its longest element.
   (a) Prove that $s_1s_2s_1s_3s_2s_1s_4s_3s_2s_1 \ldots s_{n-1}s_{n-2}\ldots s_2s_1$ is a reduced word for w_0.
   (b) What is the smallest number of generators that one can remove from this reduced word to obtain a word for the identity?
2. Let $(S_n)^J$ be a parabolic quotient of S_n modulo a set of generators J with $|J| = n-2$. Prove that, if $x, y \in (S_n)^J$, then
$$x \leq y \text{ in the Bruhat order} \qquad \Longleftrightarrow \qquad x \leq_L y \text{ in the left weak order.}$$
3. Let W be a finite Coxeter group with long element w_0 and let $w \in W$. Show that the set
$$\{x \in W \mid x \wedge w = e \text{ and } x \vee w = w_0\}$$
is an interval in the weak order.
4. Prove that the weak order of the symmetric group S_n, when considered as a graph, is isomorphic to the graph formed by the vertices and the edges of the *permutahedron* Π_n:
$$\Pi_n = \text{convex hull}\{(\pi_1, \ldots, \pi_n) \,|\, \pi \in S_n\}.$$
5. Give an example of an infinite Coxeter group whose weak order has no infinite antichains, and an example of an infinite Coxeter group whose weak order has an infinite antichain.
6. In Homework 2 you showed that the Coxeter group $\widetilde{A}_2$ whose Coxeter diagram is an unlabelled triangle can be understood in terms of the infinite equilateral triangular grid. Explain the relationship between this realization and the geometric representation of $\widetilde{A}_2$ in $\mathbb{R}^3$.
7. *(Recommended.)* Given a cube of sidelength $\frac{1}{\sqrt{2}}$, consider the 12 vectors that go from the center of the cube to the midpoints of its 12 edges. Prove that these 12 vectors form a root system for the symmetric group S_4. Draw a diagram of the cube, labeling the six positive roots with the six reflections in S_4. In a second cube draw the reflecting hyperplanes for s_1, s_2 and s_3. Find $s_1s_2s_3s_1s_2s_3(\alpha_1)$, showing the successive images of α_1 after each reflection.
8. Prove the following statement for the dihedral groups $I_2(2m)$ and $I_2(\infty)$:
   For $w \in W$ and $s_i \in S$, $\ell(ws_i) > \ell(w)$ implies $w\alpha_i > 0$, and $\ell(ws_i) < \ell(w)$ implies $w\alpha_i < 0$.
9. *(Recommended.)* Given a reduced expression $w = s_1 \ldots s_r$ in a Coxeter system (W, S), let α_i be the simple root corresponding to s_i. Prove that the positive roots that w sends to negative roots are precisely the r roots of the form $\beta_i = s_r s_{r-1} \ldots s_{i+1}(\alpha_i)$.
10. Suppose an element w of W acts by a geometric reflection in the geometric representation of $GL(V)$, in the sense that there exists a unit vector $\alpha \in V$ such that $wv = v - 2\langle v, \alpha\rangle\alpha$ for all $v \in V$. Prove that α is a root and w is a reflection (*i.e.* a conjugate of a generator) in W.