coxeter groups

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homework three

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(On paper at 10am, or as a single .pdf file by midnight, SF time.)

Instructions. Turn in your **five** best problems out of the following list. You are encouraged to work together on the homework, but you must state who you worked with on each problem. You *must* write your solutions independently and in your own words. I encourage you to type your solutions using LaTeX; that will be good practice for the final project, where LaTeX is mandatory.

1. If w_0 is an element of a Coxeter group such that $sw_0 < w_0$ for every simple reflection $s \in S$, prove that $w \leq w_0$ for all $w \in W$.
2. Consider the Coxeter system $(S_n, \{s_1, \ldots, s_{n-1}\})$. Let $J = \{s_1, \ldots, \widehat{s_k}, \ldots, s_{n-1}\}$. Describe combinatorially the elements of the parabolic quotient S_n^J, and the projection map $w \mapsto w^J$.
3. Prove that the parabolic subgroups of the Coxeter system $(S_n, \{s_1, \ldots, s_{n-1}\})$ are Young subgroups of the symmetric group S_n. Conversely, prove that every Young subgroup of S_n is a parabolic subgroup for some choice of generators.

   (Note. Given a partition A of $[n]$ as a disjoint union of sets $A_1, \ldots, A_k$, the *Young subgroup* $S_A = S_{A_1} \times \cdots S_{A_k}$ of S_n consists of the permutations of $[n]$ which map elements of A_i to elements of A_i for each i.)
4. Consider the Coxeter system $(S_4, \{s_1, s_2, s_3\})$; let $J = \{s_1, s_3\}$. Draw the Bruhat order of S_4 and the product poset $(S_4)^J \times (S_4)_J$ inside it.

   (Note. The product $P \times Q$ of two posets P and Q consists of the elements (p, q) with $p \in P, q \in Q$, ordered by declaring that $(p, q) \leq (p', q')$ if $p \leq p'$ and $q \leq q'$.)
5. Give a combinatorial proof of the following fact: in any interval $[u, v]$ in the Bruhat order of a Coxeter group, the number of elements of even rank equals the number of elements of odd rank.
6. Given a Coxeter system (W, S) and $I, J \subseteq S$, prove that
   (a) $W_I \cap W_J = W_{I \cap J}$
   (b) $\langle W_I \cup W_J \rangle = W_{I \cup J}$
   (c) $W_I = W_J$ if and only if $I = J$.
7. Let (W, S) be a finite irreducible Coxeter system and $J \subset S$. Show that if W^J is a lattice, then it is a distributive lattice and $|J| = |S| - 1$.

   (Note. A Coxeter system is finite if its Coxeter diagram is connected. A lattice L is *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in L$.)
8. Show that if one is given the Bruhat order of a Coxeter system (W, S), one can determine its Bruhat graph.
9. Let (W, S) be a Coxeter system with S finite. Show that every antichain in the Bruhat order of W is finite.

   (Note. An *antichain* of a poset is a subset of pairwise incomparable elements.)
10. Suppose W is infinite but every parabolic subgroup of W is finite. Let $W(q) = \sum_{w \in W} q^{\ell(w)}$. Prove that $W(1/q) = \pm W(q)$ as rational functions.