coxeter groups

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homework two

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(On paper at 10am, or as a single .pdf file by midnight, SF time.)

Instructions. Turn in your **five** best problems out of the following list. You are encouraged to work together on the homework, but you must state who you worked with on each problem. You *must* write your solutions independently and in your own words. I encourage you to type your solutions using LaTeX; that will be good practice for the final project, where LaTeX is mandatory.

1. Prove that the group of symmetries of an infinite equilateral triangular grid equals the Coxeter group whose Coxeter diagram is a triangle with no labels on the edges.
2. Prove that the length function $\ell : W \to \mathbb{N}$ of a Coxeter system (W, S) satisfies the following properties:
   (a) $\ell(uv)$ and $\ell(u) + \ell(v)$ have the same parity for $u, v \in W$.
   (b) $\ell(w^{-1}) = \ell(w)$ for $w \in W$.
   (c) $\ell(sw) = \ell(w) \pm 1$ for $w \in W$ and $s \in S$.
   (d) The distance function $d(u, v) = \ell(uv^{-1})$ is a metric on W.
3. Prove that $\ell(uv) = \ell(u) + \ell(v)$ if and only if there does not exist a reflection $t \in T$ such that $\ell(ut) < \ell(u)$ and $\ell(tv) < \ell(v)$.
4. Prove that any two reduced words for $w \in W$ can be obtained from each other by successively replacing $ss'ss' \cdots$ by $s'ss's \cdots$ (both of length $m(s, s')$) for $s, s' \in S$.
5. Prove that if $s_1, \ldots, s_k \in S$ are such that $s_1 \cdots s_k = e$, then one can obtain the empty word from $s_1 \cdots s_k$ by successively replacing $ss'ss' \cdots$ by $s'ss's \cdots$ (both of length $m(s, s')$) for $s, s' \in S$, and removing ss for $s \in S$. Use this to describe an algorithm for the *word problem* for Coxeter groups: determining whether a word in the generators equals the identity.
6. Are 283457619 and 298563471 comparable in the Bruhat order of S_9?
7. Let $E_\bullet$ and $F_\bullet$ be two complete flags in $\mathbb{R}^n$. Prove that the table $\{\dim(E_i \cap F_j)\}_{1 \leq i,j \leq n}$ is the rank table of a permutation in S_n.
8. Prove that the set of permutaitons $\{x \in S_{2n} \, : \, |x(i) - i| \leq n \text{ for } 1 \leq i \leq 2n\}$ forms an interval in the Bruhat order of S_{2n}. How many atoms and coatoms does this interval have?
9. An $n \times n$ matrix is *totally non-negative* if all of its minors are non-negative. Let u and v be permutations in S_n. Prove that $u \leq v$ in the Bruhat order if and only if
$$a_{1,u(1)} a_{2,u(2)} \cdots a_{n,u(n)} \geq a_{1,v(1)} a_{2,v(2)} \cdots a_{n,v(n)}$$
   for any totally non-negative matrix $(a_{ij})_{1 \leq i,j \leq n}$.
10. A permutation $w \in S_n$ is said to be *321-avoiding* if there are no $1 \leq a < b < c \leq n$ with $w(a) > w(b) > w(c)$. Prove that a permutation is 321-avoiding if and only if no reduced decomposition of w contains a subsequence of the form $s_i s_{i+1} s_i$ or $s_{i+1} s_i s_{i+1}$.