coxeter groups

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homework one

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(On paper at 10am, or as a single .pdf file by midnight, SF time.)

Instructions. Turn in your **five** best problems out of the following list. You are encouraged to work together on the homework, but you must state who you worked with on each problem. You *must* write your solutions independently and in your own words. I encourage you to type your solutions using LaTeX; that will be good practice for the final project, where LaTeX is mandatory.

1. Suppose that the Coxeter diagram of $(W, \{a, b, c\})$ has an unlabelled edge between a and b, an edge labelled m between b and c, and no edge between a and c. If the relation $bcbcacababcacbcabacbabacbc = 1$ holds, determine the value of m.
2. Show that there exist Coxeter systems (W, S) and (W', S') with $|S| \neq |S'|$ such that $W \cong W'$. (Hint. There is an example with $W \cong W' \cong D_6$.)
3. (Recommended.) Let $m : S \times S \to \{1, 2, \ldots, \infty\}$ be a Coxeter matrix. The Coxeter group associated to m is $W \cong F/N$, where F is the free group generated by N and N is the normal subgroup generated by $\{(ss')^{m(s,s')} : s, s' \in S\}$.
   (a) Let S^* be the set of finite words in the alphabet S, and define the product of two words to be their concatenation. Consider the equivalence the relation $uv \sim u(ss')^{m(s,s')}v$ for all $u, v \in S^*, s, s' \in S$. Prove that the product on S^* extends to a product on $S^*/\sim$, and that this product makes $S^*/\sim$ into a group isomorphic to W.
   (b) Prove that W is the "universal group" satisfying the Coxeter relations: If G is a group and $\{f(s)\}_{s \in S}$ are elements of G such that $(f(s)f(s'))^{m(s,s')} = 1$ for all $s, s' \in S$, then f extends uniquely to a group homomorphism $f : W \to G$.
4. Prove that every dihedral group is a Coxeter group.
5. (Recommended.) For $1 \leq i \leq n-1$ let $s_i = (i, i+1)$ be the ith simple transposition in the symmetric group S_n. Prove that $(S_n, \{s_1, \ldots, s_{n-1}\})$ is a Coxeter system.
6. Prove that the hyperoctahedral group is a Coxeter group.
7. Identify the set T of transpositions $\{(ij) : 1 \leq i < j \leq n\}$ in the symmetric group S_n with the edges of the complete graph K_n.
   (a) Prove that $A \subseteq T$ generates S_n if and only if A connects all vertices of K_n.
   (b) Prove that $A \subseteq T$ is a system of Coxeter generators for S_n if and only if A is a path connecting all vertices of K_n.
8. If a Coxeter diagram G decomposes into connected components $G_1, \ldots, G_k$, prove that the Coxeter group W of G is isomorphic to the direct product of the Coxeter groups $W_1, \ldots, W_k$ corresponding to $G_1, \ldots, G_k$.
9. Prove that any reflection in a Coxeter group has a palyndromic reduced expression.
10. Let $(W, \{s_1, \ldots, s_n\})$ be a Coxeter group. Prove that the alternating group of W is generated by $s_1s_n, s_2s_n, \ldots, s_{n-1}s_n$.