
combinatorial commutative algebra . 2009

final projects

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due dates.

- o Wednesday, April 8: rough outline. (1-2 pages)
- o Monday, May 18: final project. (~10 pages in LaTeX, 11pt, single space)

For the final project of the course, you will develop a solid understanding of a particular aspect of combinatorial commutative algebra (CCA) that interests you. You may, for instance:

- Understand the background and significance of an open problem in CCA, and solve it, or achieve some partial progress.
- Understand the current state of the art in a branch of CCA, and present it in a clear, concise, and useful survey.
- Find a new way of thinking about or proving a known result.
- Write a computer program that will be useful to researchers in CCA.

Below are some possible topics for a final project, in no particular order. The list is not comprehensive, and probably biased towards topics that I like. You may choose your own topic, but you'll need me to approve it before you start working on it. I am flexible about the topics that you choose, but you must prove to me that you learned a lot of mathematics related to CCA!

In your proposal you will describe your concrete plan of action and I will offer feedback.

some suggested sources.

The existing literature on CCA is large, deep, and broad; there are many topics within your reach. Many papers and books contain interesting open problems which you can understand and think about. Search on Google Scholar, the math arXiv, and the American Math. Society's mathscinet.

Many of the suggested texts for the course have exercises and comments which provide good project directions. Some concrete project suggestions and open problems are in:

1. Cox, Little, O'Shea. Ideals, varieties, and algorithms. Appendix D.
2. D. Eisenbud. Commutative algebra with a view towards algebraic geometry, particularly Sections 15.10.9 (open algorithmic questions), 15.12 (computer algebra projects).
3. R. Stanley. Combinatorics and commutative algebra, p. 135-143.
4. R. Stanley. Positivity problems and conjectures in algebraic combinatorics.
<http://www-math.mit.edu/~rstan/papers/problems.pdf>

some suggested general topics.

You should be able to find the references below on google, the arxiv, or mathscinet. You may need to be creative if you need access to a book that your library doesn't have. Let me know if you have looked carefully and still can't find some of the references I mention.

1. **The g -theorem and the Upper Bound Conjecture.** The f -vector of a polytope keeps track of the number f_i of i -dimensional faces. The g -theorem characterizes all possible f -vectors of simplicial polytopes. Stanley proved this conjecture using tools from commutative algebra. Several extensions, related results, and conjectures that followed.
T.Hibi. Algebraic combinatorics on convex polytopes.
R. Stanley. Combinatorics and commutative algebra.
2. **The cd -index.** A poset is Eulerian if it satisfies a condition that makes it look like the face poset of a polytope. Some of the structure of an Eulerian poset is elegantly encoded in its cd -index, which has nice properties.
R. Stanley. Combinatorics and commutative algebra.
Billera, L. J., R. Ehrenborg and M. Readdy, The cd -index of zonotopes and arrangements.
<http://www.math.cornell.edu/~billera/papers/cdzon.ps>
3. **Shellability.** Shellability is a combinatorial condition on a simplicial complex which implies many nice algebraic properties. Many nice families of combinatorial simplicial complexes are known or conjectured to be shellable.
R. Stanley. Combinatorics and commutative algebra.
4. **Characterizations of Hilbert functions.** What are the possible Hilbert functions of a graded ring? Macaulay gave a beautiful characterization for rings satisfying certain mild conditions. There are many subsequent variants, strengthenings, and related conjectures.
Stanley. Combinatorics and commutative algebra.
G. Kalai. Algebraic Shifting. <http://www.ma.huji.ac.il/~kalai/japan.ps>
5. **Ehrhart theory.** Given a lattice polytope P , the number $E_P(n)$ of integer points in the scaled polytope nP is given by a polynomial in n called the Ehrhart polynomial. This polynomial has close ties to CCA.
M. Hochster. Rings of Invariants of Tori, Cohen-Macaulay Rings Generated by Monomials, and Polytopes.
Stanley. Combinatorics and commutative algebra.
6. **Splines on simplicial complexes.** A spline on a simplicial complex Δ is a continuous function on Δ which is polynomial on each face, and is differentiable to a specified order. Applications include numerical analysis and computer graphics.
L. Billera. Homology of smooth splines: generic triangulations and a conjecture of Strang.
L. Billera and L. Rose. Gröbner basis methods for multivariate splines.
R. Stanley. Combinatorics and commutative algebra.

7. **Box splines and systems of linear equations.** Dahmen and Michelli, among many others, showed how the theory of box splines in approximation theory can be applied to study the space of nonnegative integer solutions to a system of linear equations.
Dahmen-Michelli, On the number of solutions to systems of linear diophantine equations and multivariate splines.
C. De Concini, C. Procesi. The algebra of the box spline. arXiv:0602.5019
O. Holtz and A. Ron. Zonotopal Algebra. arXiv:0708.2632
8. **Magic squares.** Let $H_n(r)$ be the number of $n \times n$ \mathbb{N} -matrices whose row sums and column sums are equal to r . Stanley and Jia used the CCA approach to Ehrhart theory and box spline theory to study this function, and offer some related open problems.
Rong-Qing Jia. Multivariate discrete splines and linear diophantine equations.
R. Stanley. Combinatorics and commutative algebra
9. **Box splines and index calculations.** De Concini, Procesi, and Vergne generalized aspects of box spline theory in order to perform computations in the index theory of elliptic operators.
C. De Concini, C. Procesi, M. Vergne. Vector partition function and generalized Dahmen-Micchelli spaces. arXiv:0805.2907
C. De Concini, C. Procesi, M. Vergne. Vector partition functions and index of transversally elliptic operators. arXiv:0808.2545
10. **Power ideals, fat point ideals, Cox rings.** A point configuration in a vector space determines several algebraic objects with beautiful combinatorial structure.
F. Ardila and A. Postnikov. Combinatorics and geometry of power ideals. arXiv:0809.2143
Geramita and Schenck. Fat Points, Inverse Systems, and Piecewise Polynomial Functions.
B. Harbourne. Problems and Progress: A survey on fat points in \mathbb{P}^2 .
B. Sturmfels and Z. Xu. Sagbi bases of Cox-Nagata rings. <http://arxiv.org/abs/0803.0892>
11. **Topology of hyperplane arrangements.** Many topological and algebraic properties of hyperplane arrangements can be understood in terms of their combinatorics.
Orlik, Terao. Arrangements of hyperplanes.
12. **Schubert calculus.** The Grassmannian variety, which is the space of k -dimensional subspaces of an n -dimensional space, can be stratified into “Schubert varieties”. This construction is useful in topology, representation theory, enumerative algebraic geometry, and symmetric functions, among others.
Miller and Sturmfels. Combinatorial commutative algebra.
Fulton. Young tableaux.
Manivel. Symmetric functions, schubert polynomials and degeneracy loci.

13. **Gröbner bases and polytopes.** An ideal I has different initial ideals with respect to different term orders. Study the Grobner fan of an ideal I , a geometric object which controls these initial ideals.
Sturmfels. Grobner bases and convex polytopes.
14. **Triangulations of polytopes and toric ideals.** There is a correspondence between initial ideals of a toric ideal and the subdivisions of a polytope. The secondary polytope of a polytope has faces corresponding to its (regular) subdivisions. The toric Hilbert scheme is the parameter space of ideals with the same Hilbert function as a given toric ideal, and it can be analyzed in terms of the triangulations of a polytope.
D. Maclagan and R. Thomas <http://arxiv.org/pdf/math.AG/9912014.pdf>
Sturmfels. Grobner bases and convex polytopes.
15. **Systems of polynomial equations.** There are nice connections between a system of polynomial equations and the combinatorics of the corresponding Newton polytope, such as Bernstein's theorem and Khovanskii's theorem on systems of equations with few monomials.
B. Sturmfels. Solving systems of polynomial equations.
16. **Applications of polynomial equations.** One can use the tools we've learned to study several polynomial systems of equations arising in economics, statistics, and phylogenetics.
B. Sturmfels. Solving systems of polynomial equations.
M. Drton, B. Sturmfels, S. Sullivan. Lectures on Algebraic Statistics
L. Pachter and B. Sturmfels. Algebraic Statistics for Computational Biology,
17. **More applications of polynomial equations.** Other applications include motion planning for robots, and algorithms for automatically proving theorems in Euclidean geometry.
Cox, Little, O'Shea. Ideals, varieties, and algorithms.
18. **Tropical geometry.** Tropical geometry studies algebraic varieties by "tropicalizing" them into polyhedral complexes that retain some of their structure.
J. Richter-Gebert, B. Sturmfels, T. Theobald. First steps in tropical geometry.
D. Maclagan and B. Sturmfels. Introduction to Tropical Geometry. (draft, online.)
19. **Invariant theory.** When a group acts on a polynomial ring, it is of interest to understand the subring of polynomials invariant under the action. Many results in algebraic geometry and commutative algebra were driven by the goal to understand this setup.
Cox, Little, O'Shea. Ideals, varieties, and algorithms.
Kane. Reflection groups and invariant theory.
Kraft, Procesi. Classical invariant theory. <http://www.math.unibas.ch/~kraft/Papers/KP-Primer.pdf>
Sturmfels. Algorithms in invariant theory.

20. **Cluster algebras.** A cluster algebra is a commutative ring with a set of generators grouped into clusters which satisfy certain properties. They are defined in an elementary way and have deep connections to many fields.
S. Fomin's cluster algebra portal: <http://www.math.lsa.umich.edu/~fomin/cluster.html>
21. **Topological combinatorics of posets.** Explore the topological approach to poset combinatorics, focusing for example on the family of Cohen-Macaulay posets.
A. Björner, A.M. Garsia, and R.P. Stanley, An introduction to Cohen-Macaulay partially ordered sets. M. Wachs. Poset topology: Tools and applications. arXiv:0602226
22. **Resolutions of edge ideals.** The edge ideal of a graph G has a generator $x_i x_j$ for each edge ij of the graph. From the invariants of its minimal resolution one can recover information about G .
R. Villarreal. Monomial algebras.
http://flash.lakeheadu.ca/~avantuy1/papers/fields_2006_tutorial.pdf

some suggested recent papers.

23. A. Tchernev. Representations of matroids and free resolutions for multigraded modules.
<http://front.math.ucdavis.edu/0409.5591>
24. Floystad. The colorful Helly theorem and colorful resolutions of ideals.
<http://arxiv.org/pdf/0809.3366.pdf>
25. Suyoung Choi, Jang Soo Kim, A combinatorial proof of a formula for Betti numbers of a stacked polytope. <http://arxiv.org/abs/0902.2444v1>
26. Ezra Miller, Topological Cohen-Macaulay criteria for monomial ideals.
<http://arxiv.org/abs/0809.1458>
27. Alin Stefan, Classifications of Cohen-Macaulay modules - The base ring associated to a transversal polymatroid.
<http://arxiv.org/abs/0808.0252>
28. T. Kyle Petersen, Pavlo Pylyavskyy, David E Speyer, A non-crossing standard monomial theory.
<http://arxiv.org/abs/0806.1776>
29. Uwe Nagel, Sonja Petrovic, Properties of cut ideals associated to ring graphs.
<http://arxiv.org/abs/0806.0585>
30. Alicia Dickenstein, Laura Felicia Matusevich, Ezra Miller, Combinatorics of binomial primary decomposition.
<http://arxiv.org/abs/0803.3846>
31. Harm Derksen, Symmetric and Quasi-Symmetric Functions associated to Polymatroids.
<http://arxiv.org/abs/0801.4393>
32. Francesco Brenti, Volkmar Welker, The Veronese Construction for Formal Power Series and Graded Algebras.
<http://arxiv.org/abs/0712.2645>
33. Uwe Nagel, Victor Reiner, Betti numbers of monomial ideals and shifted skew shapes.
<http://arxiv.org/abs/0712.2537>
34. Gunnar Floystad, Cellular resolutions of Cohen-Macaulay monomial quotient ring.
<http://arxiv.org/abs/0710.3244>