Are two ideals equal? (An application)

To test whether $I = J$, choose any monomial order $<$ and compute the reduced Gröbner bases G and H for I and J. $\quad I = J \iff G = H$

$$I = J \Rightarrow G = H \quad \text{(by thm above)}$$

$$G = H \Rightarrow I = \langle G \rangle = \langle H \rangle = J$$

Ex. 2) $I = \langle x^3y - xy^2 + 1,\ x^2y^2 - y^3 - 1 \rangle$

$J = \langle xy^3 + y^3 + 1,\ x^3y - x^3 + 1,\ x + y \rangle$

For $<\ =$ lex with $x > y$,

$$G = H = \{x + y,\ y^4 - y^3 + 1\}$$

so $I = J$.

Ex 1.) lex., $x > y$ for:

$$I = \langle \underbrace{x^2 + xy^5 + y^4}_{h_1},\ \underbrace{xy^6 - xy^3 + y^5 - y^2}_{h_2},\ \underbrace{xy^5 - xy^2}_{h_3} \rangle$$

$S(h_1, h_2) \equiv S(h_1, h_3) \equiv 0 \mod \{h_1, h_2, h_3\}$

$S(h_2, h_3) \equiv \underbrace{y^5 - y^2}_{h_4} \mod \{h_1, h_2, h_3\}$

$S(h_1, h_4) \equiv S(h_2, h_4) \equiv S(h_3, h_4) \equiv 0 \mod \{h_1, h_2, h_3, h_4\}$

So

$$\{x^2 + xy^5 + y^4,\ xy^6 - xy^3 + y^5 - y^2,\ xy^5 - xy^2,\ y^5 - y^2\}$$

is a Gröbner basis. Then

$$\{x^2 + xy^5 + y^4,\ y^5 - y^2\}$$

is a minimal Gröbner basis. Now

$$x^2 + xy^5 + y^4 \equiv x^2 + xy^2 + y^4 \mod y^5 - y^2$$

So

$\boxed{\{x^2 + xy^2 + y^4,\ y^5 - y^2\} \text{ is the reduced Gröbner basis.}}$

Elimination Theory: (Solving Systems of Polynomial Equations)

(An application.)

Ex: $\begin{cases} 2x^2 + 2xy + y^2 - 2x - 2y = 0 & \text{ellipse} \\ x^2 + y^2 = 1 & \text{circle} \end{cases}$

Clever manipulation: $5y^4 - 4y^3 = 0$

$y = 0 \to x = 1 \quad$ or $\quad y = 4/5 \to x = -3/5$

How to do this in general?

Same idea:
1. Look for $p(x_n) = 0$, solve for x_n
2. Look for $q(x_{n-1}, x_n) = 0$, solve for x_{n-1} for each sol. a_n in 1.
3. Look for $r(x_{n-2}, x_{n-1}, x_n) = 0$, solve for x_{n-2} for each sol a_{n-1} in 2, a_n in 1

⋮

This "amounts" to computing the <u>elimination ideals</u>

$$I_i = I \cap \mathbb{F}[x_{i+1}, \dots, x_n]$$

Theorem Let $G = \{g_1, \dots, g_m\}$ be a G.b. for I wrt the lex order $x_1 > \dots > x_n$, and let

$$G_i = G \cap \mathbb{F}[x_{i+1}, \dots, x_n]$$

Then G_i is a G.b. for I_i. (wrt lex, $x_{i+1} > \dots > x_n$)

So simple!

In particular, $I_i \neq 0 \iff G_i \neq \emptyset$

desirable for elimination!

Pf Need: $\mathrm{in}(I_i) = \langle \mathrm{in}(G_i) \rangle$

Let $f \in I_i$. Since G is a G.b.,

$$\mathrm{in}(f) = a_1\,\mathrm{in}(g_1) + \dots + a_m\,\mathrm{in}(g_m)$$

involves only $x_{i+1}, \dots, x_n$ — delete all monomials involving $x_1, \dots, x_i$

$$\mathrm{in}(f) = \alpha_a\,\mathrm{in}(g_a) + \dots + \alpha_c\,\mathrm{in}(g_c)$$

$\mathrm{in}(f) \in \langle \mathrm{in}(g_a), \dots, \mathrm{in}(g_c) \rangle$ as an ideal in $\mathbb{F}[x_{i+1}, \dots, x_n]$

But in lex order, if $\mathrm{in}(g_a)$ involves only $x_{i+1}, \dots, x_n$ then g_a involves only $x_{i+1}, \dots, x_n$ so $g_a \in G_i$

So $\mathrm{in}(f) \in \langle \mathrm{in}(G_i) \rangle$ ∎

<u>Computing $I \cap J$:</u>

(An application)

If $I = \langle f_1, \dots, f_a \rangle$ and $J = \langle g_1, \dots, g_b \rangle$ then

$I + J = \langle f_1, \dots, f_a, g_1, \dots, g_b \rangle$

$IJ = \langle f_1 g_1, \dots, f_1 g_b, \dots, f_a g_1, \dots, f_a g_b \rangle$

$I \cap J = ?$

Prop. a) $tI + (1-t)J$ is an ideal in $\mathbb{F}[t, x_1, \dots, x_n]$

b) $I \cap J = \big(tI + (1-t)J\big) \cap \mathbb{F}[x_1, \dots, x_n]$

So $I \cap J$ is the first elim. ideal of $tI + (1-t)J$, w.r.t. $t > x_1 > \dots > x_n$, and we can compute it!

Pf. a) clear

b) $\subseteq$: clear

$\supseteq$: Let $f = tf_1 + (1-t)f_2$

$f \in \mathbb{F}[x_1, \dots, x_n]$, $f_1 \in I$, $f_2 \in J$

Plugging in $t = 0$, we get $f = f_2$

$\Rightarrow f = f_1$

$\Rightarrow f \in I \cap J$.