Idea of Buchberger: Can I generate missing initial monomials using combins. of $g_1, \dots, g_m$?
It suffices to check the "basic" combinations $S(g_i, g_j)$:

How do you construct a Gröbner basis?

Buchberger's Algorithm

Input: $<$, I, $F = \{f_1, \dots, f_n\}$ generating I

Output: A Gröbner basis G of I (containing F)

Let $\underset{\substack{\uparrow \\ \text{Gröbner basis}}}{G := F}$ $\qquad \underset{\substack{\uparrow \\ \text{"unchecked" pairs}}}{B := \binom{F}{2}}$

While $B \neq 0$:

$\quad$ Pick $\{f, g\} \in B$

$\quad$ Let $r = S(f, g) \bmod G$.

$\quad$ If $r \neq 0$ then

$\quad\quad$ $G := G \cup \{r\}$

$\quad\quad$ $B := B \cup \{\{r, h\} \mid h \in G, h \neq r\}$

$\quad$ $B := B \setminus \{f, g\}$

I.e.: Check all pairs $\{f, g\}$ in G:
- If $S(f, g) = 0 \bmod G$, ok. Go to next pair
- If $S(f, g) = r \neq 0 \bmod G$, add r to S

Repeat until all pairs are ok.

A Gröbner basis $\{g_1, \dots, g_m\}$ is <u>minimal</u> if
- each $in(g_i)$ is monic
- no $in(g_j)$ is a multiple of $in(g_i)$ $(i \neq j)$

It is <u>reduced</u> if
- each $in(g_i)$ is monic
- no term of g_j is a multiple of $in(g_i)$ $(i \neq j)$

Theorem Given $I, <$, there is a <u>unique</u> reduced Gröbner basis.

Pf. <u>Existence</u>: Start with any Gröbner basis.

1. make each $in(g_i)$ monic
2. remove any unnecessary $in(g_i)$
3. divide each g_i by $g_1, \dots, \hat{g_i}, \dots, g_k$ and let the remainder be r_i.

$\qquad$ $in(g_i)$ is not a multiple of $in(g_j)$ $(j \neq i)$
$\qquad$ so it also occurs in $r_i \Rightarrow in(g_i) = in(r_i)$

So $\{r_1, \dots, r_k\}$ is a reduced Gröbner basis.

<u>Uniqueness</u>: Sup $G = \{g_1, \dots, g_m\}$ and $G' = \{g_1', \dots, g_m'\}$
(any two minl Gröbner bases for I have the same size and leading terms, by HW2).
Say $in(g_i) = in(g_i') = h_i$. Let $f_i = g_i - g_i' \in I$.
$in(f_i) \in in(I) \Rightarrow$ some $\underset{= in(g_j)}{in(g_j)} \mid \underset{\text{a term in } g_i \text{ or } g_i'}{in(f_i)}$

$\Rightarrow f_i = 0 \Rightarrow g_i = g_i'$ ∎