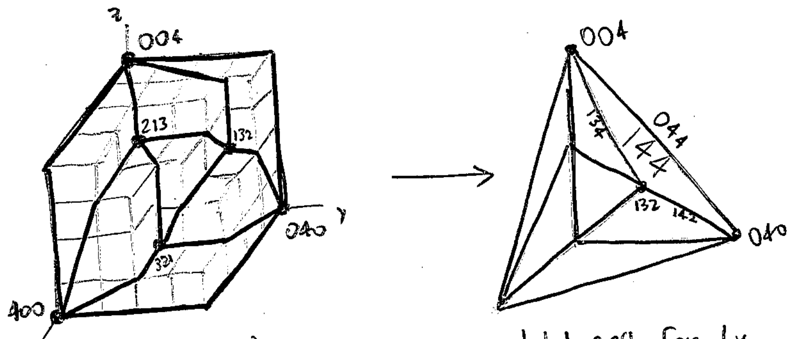
The 3-D case

$I = \langle x^4, y^4, z^4, x^3y^2z, xy^3z^2, x^2yz^3 \rangle$

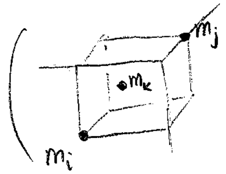


Buch(I)

label each face by lcm (vertices)

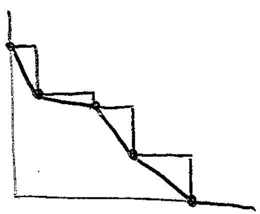
Def The <u>Buchberger graph</u> Buch(I) of a monomial ideal $I = \langle m_1, \dots, m_r \rangle$ has vertices $1, \dots, r$ and

$\begin{pmatrix} ij \text{ is} \\ \text{an edge} \end{pmatrix} \iff$ there is no k with
- $m_k \mid \mathrm{lcm}(m_i, m_j)$
- $\deg_{(x_a)} m_k < \deg_{(x_a)} \mathrm{lcm}(m_i, m_j)$ for all $x_a \mid \mathrm{lcm}(m_i, m_j)$



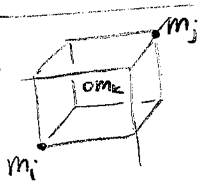
(is what one shouldn't have)

Ex In 2-D,



Prop. Syz(I) is generated by the basic syzygies $S(m_i, m_j)$ with ij an edge of Buch(I)

Pf. If:



then $S(m_i, m_j)$ can be written in term of $S(m_i, m_k)$ and $S(m_k, m_j)$, via

$$0 = \frac{m_{ijk}}{m_{ij}} S(m_i, m_j) + \frac{m_{ijk}}{m_{jk}} S(m_j, m_k) + \frac{m_{ijk}}{m_{ik}} S(m_k, m_i)$$

∎

(Ex: $m_i = 400$, $m_j = 132$, $m_k = 321$)

- This graph almost always embeds nicely:

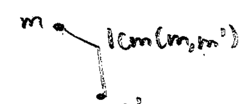
Def I is strongly generic if

$\begin{pmatrix} \text{if } x^i y^j z^k, x^{i'} y^{j'} z^{k'} \\ \text{are generators} \end{pmatrix} \Rightarrow \begin{matrix} (i \neq i' \text{ or } i = i' = 0) \\ (j \neq j' \text{ or } j = j' = 0) \\ (k \neq k' \text{ or } k = k' = 0) \end{matrix}$

Prop If I is strongly generic in $\mathbb{F}[x, y, z]$ then Buch(I) is a planar, connected graph, canonically embedded in the staircase surface of I. If I has gens. x^a, y^b, z^c, then this is a triangulated triangle.

<u>Sketch</u>: Key: m, m' gens $\overset{\text{genericity}}{\Rightarrow}$ $\mathrm{lcm}(m, m')$ is on the staircase surface.

So draw



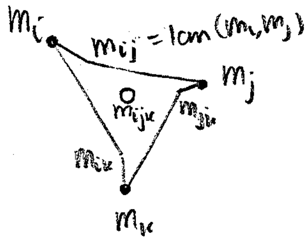
Thm If I is strongly generic in $\mathbb{F}[x,y,z]$ then the embedded $\mathrm{Buch}(I)$ gives a minl. free resol. of I.

Sketch:

- vertices of $\mathrm{Buch}(I) \longleftrightarrow$ gens. of I
- edges of $\mathrm{Buch}(I) \longleftrightarrow$ gens. of $\mathrm{Syz}(I)$
- triangles of $\mathrm{Buch}(I) \longleftrightarrow$ gens of $\mathrm{Syz}^2(I)$.

$\updownarrow$

("peaks" of I)



$\blacksquare$

In the example, **Resolution by picture**

- Minimal free resolution of J:

$$0 \to \underset{(f)}{R^7} \xrightarrow{\partial^2} \underset{(e)}{R^{12}} \xrightarrow{\partial^1} \underset{(v)}{R^6} \xrightarrow{\partial^0} I \to 0$$

- Hilb. series of J:

$$\frac{\underbrace{(x^4 + \cdots + x^2yz^3)}_{\text{vertex labels}} - \underbrace{(x^4y^4 + \cdots + xy^3z^4)}_{\text{edge labels}} + \underbrace{(x^4y^4z + \cdots + x^3y^3z^3)}_{\text{face labels}}}{(1-x)(1-y)(1-z)}$$

- Irred. decomp. of J:

$$J = \langle x^4, y^4, z \rangle \cap \cdots \cap \langle x^3, y^3, z^3 \rangle \quad \text{(face labels)}$$

What if J isn't strongly generic?

- Deform it as J_ϵ with gens shifted over by small deformations.
- Resolve J_ϵ by a planar graph G_ϵ.
- Set ϵ back to 0 to get a resol. of J.

Ex: $I = \langle x,y,z \rangle^3 = \langle x^3, x^2y, x^2z, xyz, \ldots \rangle$ in $\mathbb{F}[x,y,z]$

$\downarrow$

$I_\epsilon = \langle x^3, x^2y, x^{2.1}z, xy^{1.1}z^{1.01}, \ldots \rangle$ in $\mathbb{F}[x^{0.01}, y^{0.01}, z^{0.01}]$