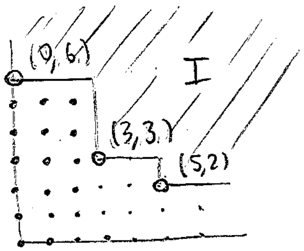
Q: Squarefree and Borel-fixed monomial ideals are nice, but what about the other monomial ideals?

$\boxed{\text{The 2-D case}}$

$I = \langle m_1, \dots, m_r \rangle = \langle x^{a_1}y^{b_1}, \dots, x^{a_r}y^{b_r} \rangle$

$\begin{pmatrix} a_1 > \dots > a_r \\ b_1 < \dots < b_r \end{pmatrix}$



basic syzygies:
$$\begin{cases} y(\underset{\substack{\uparrow \\ m_1}}{x^5y^2}) - x^2(\underset{\substack{\uparrow \\ m_2}}{x^3y^3}) = 0 \\ y^3(\underset{\substack{\uparrow \\ m_2}}{x^3y^3}) - x^3(\underset{\substack{\uparrow \\ m_3}}{x^0y^6}) = 0 \end{cases}$$

(The one between m_1, m_3 is implied)

So the minl free resolution of R/I:

$$0 \to R^2 \xrightarrow{\begin{bmatrix} y & 0 \\ -x^2 & y^3 \\ 0 & -x^3 \end{bmatrix}} R^3 \xrightarrow{[x^5y^2,\, x^3y^3,\, y^6]} R \longrightarrow R/I \to 0$$

In general,

Prop The minl free resol. of R/I in $\mathbb{F}[x,y]=R$ has the form

$$0 \to \underset{\substack{a_1b_2 \\ \vdots \\ a_{r-1}b_r}}{R^{r-1}} \to \underset{\substack{a_1b_1 \\ \vdots \\ a_rb_r}}{R^r} \to R \to R/I \to 0$$

← grading

outer corners   inner corners

Cor
$$K(R/I; x,y) = 1 - \sum_{i=1}^{r} x^{a_i}y^{b_i} + \sum_{i=1}^{r-1} x^{a_i}y^{b_{i+1}}$$

I irred: no nontrivial $I = J \cap K$

Prop I has irredundant irreducible decomp

$$I = \underset{\substack{\uparrow \\ \text{omit} \\ \text{if } b_1=0}}{\langle y^{b_1} \rangle} \cap \langle x^{a_1}, y^{b_2} \rangle \cap \langle x^{a_2}, y^{b_3} \rangle \cap \dots \cap \underset{\substack{\uparrow \\ \text{omit} \\ \text{if } a_r=0}}{\langle x^{a_r} \rangle}$$

Pf: Draw picture.

Do these "resolutions by pictures" generalize?

Yes, but we need to go to higher dim.

Two general techniques

Say $J = \langle x^4, y^4, z^4, x^3y^2z, xy^3z^2, x^2yz^3 \rangle \subset \mathbb{F}[x,y,z] = R$

① Reduce to squarefree case.

Let the "polarization" of J be

$$I = \langle x_1x_2x_3x_4, y_1y_2y_3y_4, z_1z_2z_3z_4, x_1x_2x_3y_1y_2z_1, \dots \rangle$$

$$\subset \mathbb{F}[x_1, x_2, \dots, z_3, z_4] = S$$

We know how to deal with this one

Nice facts:

- $R/J \cong (S/I) / \langle x_1 - x_2, x_2 - x_3, \dots, z_3 - z_4 \rangle$
- From the min'l free resol. / Hilbert series of S/I_Δ we get those of R/J by setting $x_i = x$, $y_i = y$, $z_i = z$

Trouble

(alg. of I) $\longleftrightarrow$ (combin/top of Δ)

huge!!

In this case the f-vec is
(1,12,66,220,492,768,832,264,51)

So this is
- good for proving theorems
- bad for actually computing.

② Reduce to Borel-fixed case

$$J \downarrow$$

$$\mathrm{gin}_{\text{revlex}}(J) = \langle x^4, x^3y, x^2y^2, \dots \rangle \quad \text{(17 gens)}$$

We know how to deal with this one

Nice facts.

- Coarse Hilbert series are equal: (change of coords)

$$K(R/J; t) = K(R/\mathrm{gin}_< J; t)$$
$$= 1 + 3t + 6t^2 + 10t^3 + 12t^4 + 12t^5 + t^6$$

- Betti numbers are bounded:

$$\beta_{i,a}(R/J) \le \beta_{i,a}(R/\mathrm{gin}\, J) \quad \text{(later thm)}$$

Trouble

Cannot use this to compute the fine Hilbert series or the min'l free resolution.

Back to pictures: