Ex

$\sigma)$

Borel-fixed monomial ideals

A very nice family of (non-squarefree) monomial ideals. ⚠ Assume char $\mathbb{F} = 0$.

$GL_n(\mathbb{F}) = \{\text{invertible } n\times n \text{ matrices}\}$ general linear group

$\cup$

$B_n(\mathbb{F}) = \{\text{upper triangular matrices}\}$ Borel subgroup

$\cup$

$T_n(\mathbb{F}) = \{\text{diagonal matrices}\}$ algebraic torus group

$GL_n(\mathbb{F})$ acts on $R = \mathbb{F}[x_1, \dots x_n]$ by "change of basis":

$$\boxed{g_{ij}}\, p(x_1 \dots x_n) = p\left(\sum_i g_{1i} x_i, \dots, \sum_i g_{ni} x_i\right)$$

$$\left(\text{So } \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} (x^2+y^2-1) = (x+2y)^2 + (x-y)^2 - 1 = 2x^2+2xy+5y^2-1.\right)$$

Who is fixed under T? B?

Prop An ideal $I \subseteq R$ is fixed under T $\iff$ It is a monomial ideal

$TI \supseteq I$ ✓
$TI \subseteq I$?

Pf $\Leftarrow$ $\underbrace{\begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{bmatrix}}_{\in T} \underbrace{(x_1^{a_1} \cdots x_n^{a_n})}_{\in I} = (t_1 x_1)^{a_1} \cdots (t_n x_n)^{a_n} \in I$

$\Rightarrow$ See [MS].

Prop An ideal $I \subseteq R$ is fixed under GL_n

$$\Updownarrow$$

$$I = \langle x_1, \dots, x_n \rangle^d \quad \text{for some } d$$

Prop An ideal $I \subseteq R$ is fixed under B

$$\Updownarrow$$

- I is monomial, and
- If $m \in I$, $x_j \mid m$, $i<j$, then $m \frac{x_i}{x_j} \in I$.

"Borel-fixed monomial ideals"

$\Rightarrow$ • Since $T \subset B$, B-fixed $\Rightarrow$ T-fixed $\Rightarrow$ monomial

• $\underbrace{x_1^{a_1} \cdots x_n^{a_n}}_{m \in I} = x_1^{a_1} \cdots (x_i + x_j)^{a_i} \cdots x_n^{a_n} \in I$

$$= x_1^{a_1} \cdots x_i^{a_i} \cdots x_n^{a_n}$$
$$+ a_i x_1^{a_1} \cdots (x_i^{a_i - 1} x_j) \cdots x_n^{a_n} \leftarrow m \frac{x_j}{x_i}$$
$$+ \cdots$$

But I is monomial so $m \frac{x_j}{x_i} \in I$.

$\Leftarrow$ Easy - see [MS]. ∎

Note To check whether I is Borel-fixed, it is enough to check condition when m is a generator.

Ex: $\langle x_1 x_2, x_2^3, x_1 x_3^3 \rangle$ no. $x_1 x_2 \to x_1 x_1$

$\langle x_1^2, x_1 x_2, x_2^3, x_1 x_3^3 \rangle$ yes.

They occur very naturally as follows:

Fix $I \subset R$, $<$ monomial order $\Rightarrow$ Get $\mathrm{in}_<(I)$.

But this may be the "wrong" answer in this sense:

We could perform a change of variables $g \in GL_n$ and compute $\mathrm{in}_<(g \cdot I)$.

Prop Most choices of $g \in GL_n$ give the same $\mathrm{in}_<(g \cdot I)$, called $\mathrm{gin}_<(I)$.

"Most": In a "Zariski open set" of GL_n: For all g not satisfying certain polynomial equations.

Ex. $I = \langle x_1^2, x_2^2 \rangle$
$<$: lex with $x_1 > x_2$

Then $\mathrm{in}_<(I) = \langle x_1^2, x_2^2 \rangle$

$\mathrm{in}_<\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} I\right) = \langle x_1^2, x_1 x_2, x_2^3 \rangle$

$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} I = \langle (x_1+x_2)^2, (x_1-x_2)^2 \rangle = \langle x_1^2 + x_2^2, x_1 x_2 \rangle$

Fact: $\mathrm{in}_<\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} I\right) = \langle x_1^2, x_1 x_2, x_2^3 \rangle$ unless $ac = 0$.

So $\boxed{\mathrm{gin}_<(I) = \langle x_1^2, x_1 x_2, x_2^3 \rangle}$

Theorem (Galligo, Bayer, Stillman)
$\mathrm{gin}_<(I)$ is Borel-fixed.