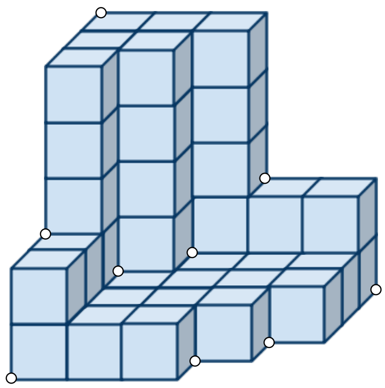
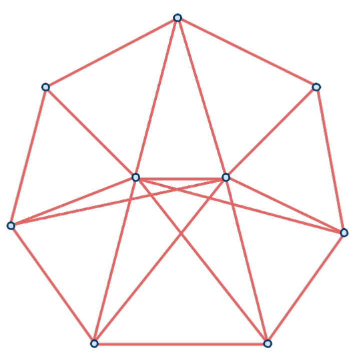
Short Homework 2

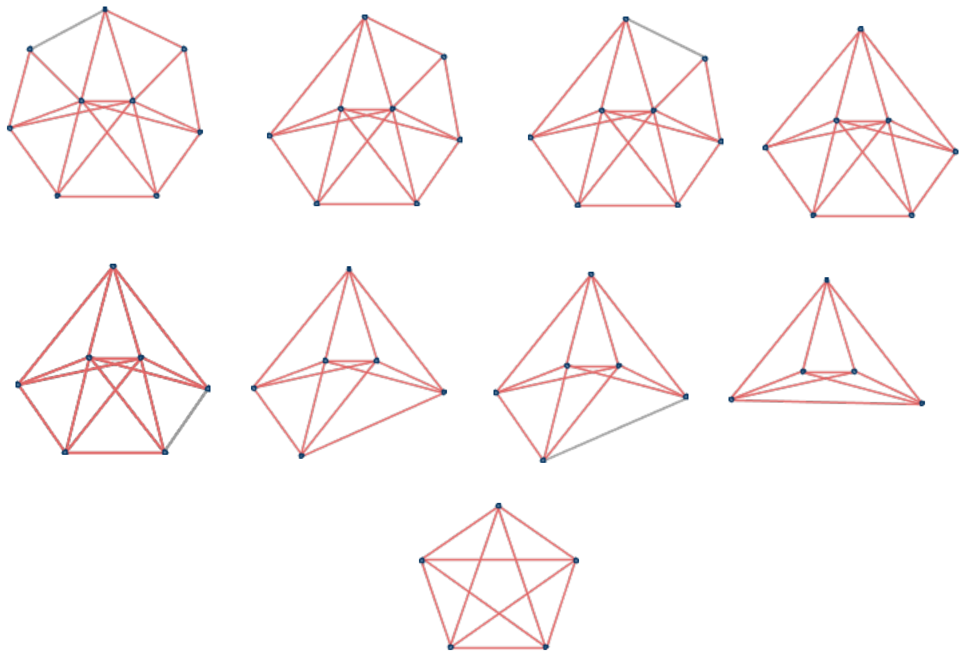
Let $I = \langle x^5, y^5, z^5, x^2yz, xy^2z, x^3z^2, y^3z^2, x^4y^3, x^3y^4 \rangle \subseteq \mathbb{F}[x, y, z]$.

(a) Draw the staircase surface for I.

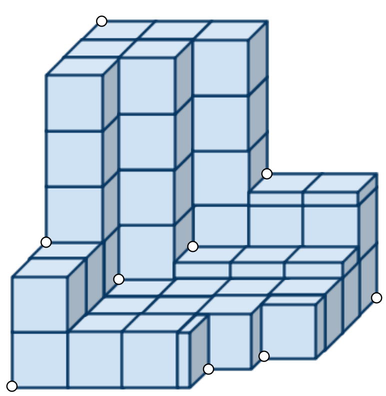


(b) In a separate picture, draw the Buchberger graph of I. Show that the complete graph K_5 can be obtained from G by repeatedly contracting some edges. Since K_5 cannot be drawn in the plane, neither can G.



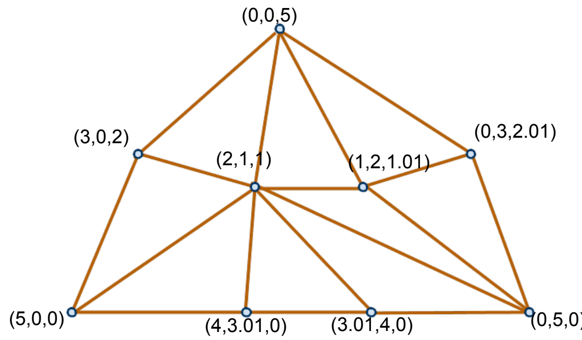


(c) Draw the staircase surface of a generic deformation J of I. (Move some of the 9 inner corners by tiny amounts to make I generic.)



(d) Use the resolution by picture of J to compute the coarse Hilbert series of I.

The Buchberger graph for J is the following:



Using this graph and forgetting about the ϵ we get the following Hilbert series (vertices-edges+faces/variables):

$$\begin{aligned}H(I;x,y,z) =&[(x^5+y^5+z^5+x^2yz+xy^2z+x^3z^2+y^3z^2+x^4y^3+x^3y^4)\\&-(x^5y^3+x^4y^4+x^3y^5+x^5z^2+x^5yz+x^4y^3z+x^3y^4z+xy^5z+x^2y^5z+\\&y^5z^2+x^3yz^2+x^2y^2z+xy^3z^2+x^3z^5+x^2yz^5+xy^2z^5+y^3z^5)\\&+(x^5yz^2+x^5y^3z+x^4y^4z+x^3y^5z+x^2y^5z+xy^5z^2+x^3yz^5+x^2y^2z^5+xy^3z^5)]\\&/(1-x)(1-y)(1-z).\end{aligned}$$

The coarse Hilbert series for I is obtained by substituting x into y and z and thus we obtain:

$$\begin{aligned}H(I;x) =&[(2x^7+5x^5+2x^4)-(10x^8+4x^7+2x^6+x^5)+(6x^9+3x^8)]/(1-x)^3\\=&[(6x^9-7x^8-2x^7-2x^6+4x^5+2x^4)]/(1-x)^3.\end{aligned}$$