Combinatorial commutative algebra short homework 1

Alex Fink

(a) We let

$$\mathbf{g} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

be our generic matrix in $GL_3(F)$, using these variable names for easy typability, so that $\mathbf{g}I = ((ax+by+cz)(dx+ey+fz), (ax+by+cz)(gx+hy+iz))$. We now run Buchberger's algorithm, and for this subproblem, I'm using Macaulay 2 to handle the polynomials, but stepping through the algorithm by hand for explicitness. For the purposes of finding the initial ideal we work in the field $\mathbb{Q}(a,b,c,d,e,f,g,h,i)$, so that some denominators are produced, and make all our Gröbner basis elements monic. Then the genericity condition we want is that the denominators that appear are nonzero.

We reduce one of these generators of $\mathbf{g}I$ mod the other to use as our Gröbner basis elements:

$$s_0 = x^2 + (bd+ae)/(ad)xy + (cd+af)/(ad)xz + (be)/(ad)y^2 + (ce+bf)/(ad)yz + (cf)/(ad)z^2$$
$$s_1 = xy + (-fg+di)/(-eg+dh)xz + (-b)/(-a)y^2 + (-ceg-bfg+cdh+bdi)/(-aeg+adh)yz + (-cfg+cdi)/(-aeg+adh)z^2$$

Next, we have a syzygy y in $s_0 - x$ in s_1. So we compute $ys_0 - xs_1$ and reduce, and the expression reduces all the way to zero; thus we're actually done already[1], and have a two-generator initial ideal $J := \operatorname{in} \mathbf{g}I = (x^2, xy)$.

(b) We have to check that multiplying a monomial $m \in J$ by some x_i/x_j with $i < j$ and $x_j \mid m$ yields a monomial still in J (where $(x_1, x_2, x_3) = (x, y, z)$). It's enough to check the cases where m is a generator. For the generator x^2 there is nothing to do, since x is greater than all the other variables. For the generator xy we need only check $xy \cdot x/y = x^2$, since x is the only variable greater than y, and indeed $x^2 \in J$. So J is Borel-fixed.

(c) We go back to the computation in (a), and consider the cases where some of the denominators do happen to be zero. In each case, for brevity, we just list the Gröbner basis in Macaulay 2 format and the associated initial ideal. (I'm now doing the entire computation with `gb` in Macaulay 2.)

For completeness, the generic Gröbner basis is

```
| xy+(fg-di)/(eg-dh)xz+b/ay2+(ceg+bfg-cdh-bdi)/(aeg-adh)yz+(cfg-cdi)/(aeg-
--------------------------------------------------------------------------
adh)z2 x2+(ceg-bfg-cdh-afh+bdi+aei)/(aeg-adh)xz-b2/a2y2+(-bceg-b2fg+bcdh-
--------------------------------------------------------------------------
abfh+b2di+abei)/(a2eg-a2dh)yz+(-bcfg-acfh+bcdi+acei)/(a2eg-a2dh)z2 |
```

[1] I didn't expect that; so I spent quite a while figuring out how I was going to make what looked like it would be a long and drawn-out computation manageable, and far less actually computing. Probably I should have expected it.

for the initial ideal (xy, x^2).

Now suppose $a = 0$, but we're otherwise generic. Then we get

```
| y2+(ceg+bfg-cdh-bdi)/(beg-bdh)yz+(cfg-cdi)/(beg-bdh)z2
------------------------------------------------------------------------
xy+c/bxz+(-fh+ei)/(eg-dh)yz+(-cfh+cei)/(beg-bdh)z2 |
```

for the initial ideal (y^2, xy).

Proceeding depth-first, we have a b in the denominator here, so suppose $a = b = 0$. Then we get

```
| yz+(fg-di)/(eg-dh)z2 xz+(-fh+ei)/(eg-dh)z2 |
```

and the initial ideal (yz, xz).

We cannot take $a = b = eg - dh = 0$, for this creates a linear dependence between the first two rows of $\mathbf{g}$. So instead, back up and take $a = eg - dh = 0$. We get

```
| yz+c/bz2 xy+c/bxz+h/gy2-c2e/b2dz2 |
```

and the initial ideal (yz, xy).

From here we can consider $a = eg - dh = g = 0$. This is a reducible condition, being the union of $a = g = d = 0$ and $a = g = h = 0$. The former of these we ignore as not yielding an invertible matrix, while the latter gives

```
| yz+c/bz2 xy+c/bxz+e/dy2-c2e/b2dz2 |
```

and the initial ideal (yz, xy) still.

We also consider $a = eg - dh = d = 0$, which is similarly reducible, so we actually take $a = d = e = 0$ and get

```
| yz+c/bz2 xy+c/bxz+h/gy2-c2h/b2gz2 |
```

and again the initial ideal (yz, xy).

This finishes the $a = 0$ case. So, returning to our generic case, take $eg - dh = 0$. This gives

```
| xz+b/ayz+c/az2 x2+(bg+ah)/agxy+be/ady2+(-bcg+ach)/a2gyz-c2/a2z2 |
```

for the initial ideal (xz, x^2).

And, since the only denominators appearing there are a which we've already considered the vanishing of, we're done, and we've listed all the initial ideals that appear.