3. a) We have

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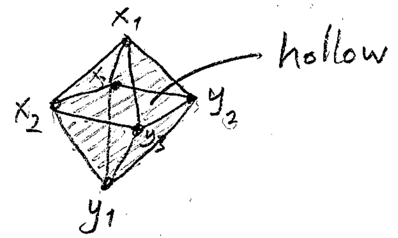
$$\Delta(I) = \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{y_1\}, \{y_2\}, \{y_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_1, y_2\}, \{x_1, y_3\}, \{x_2, x_3\}, \{x_2, y_1\}, \{x_2, y_3\}, \{x_3, y_1\}, \{x_3, y_2\}, \{x_1, x_2, x_3\}, \{x_1, x_2, y_3\}, \{x_1, x_3, y_2\}, \{x_1, y_2, y_3\}, \{x_2, x_3, y_1\}, \{x_2, y_1, y_3\}, \{x_3, y_1, y_2\}, \{y_2, y_3, y_3\}\}$$

$=$ 

b) The f-vector is $(1, 6, 12, 8)$, and by Stanley's trick the h-vector is

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 6 & & & \\ & & 1 & & 5 & & 12 & & \\ & 1 & & 4 & & 7 & & 8 & \\ 1 & & 3 & & 3 & & 1 & & \end{array}$$

$(1, 3, 3, 1)$. It is symmetric because $\Delta(I)$ is the boundary of a polytope.

c) By a theorem from class we have

$$H(R/I, x) = \frac{h_\Delta(x)}{(1-x)^d} = \frac{1 + 3x + 3x^2 + x^3}{(1-x)^3}$$

d) Using what we've seen from class, our free resolution must look like this

$$0 \longrightarrow \underset{11111}{R} \xrightarrow{\begin{array}{cc} x_1x_2x_3y_1y_2y_3 & \\ \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} & \begin{array}{l} x_1x_2y_1y_2 \\ x_1x_3y_1y_3 \\ x_2x_3y_2y_3 \end{array} \end{array}} \underset{\begin{array}{c} 110110 \\ 101101 \\ 011011 \end{array}}{R^3} \xrightarrow{\begin{array}{cc} x_1x_2y_1y_2 \quad x_1x_3y_1y_3 \quad x_2x_3y_2y_3 & \\ \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} & \begin{array}{l} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{array} \end{array}} \underset{\begin{array}{c} 100100 \\ 010010 \\ 001001 \end{array}}{R^3} \xrightarrow{\begin{array}{c} x_1y_1 \quad x_2y_2 \quad x_3y_3 \\ \begin{bmatrix} -1 & -1 & -1 \end{bmatrix} \end{array}} I \longrightarrow 0$$

e) The Betti numbers are nonzero only for degrees
000000, 100100, 010010, 001001, 110110, 101101, 011011, 111111.

So

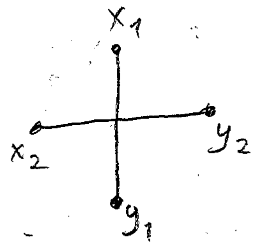
$K^{000000}(\Delta) = \cancel{\{}\emptyset\cancel{\}} = \emptyset$

$K^{100100}(\Delta) = \{\emptyset\}$

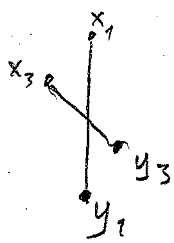
$K^{010010}(\Delta) = \{\emptyset\}$

$K^{001001}(\Delta) = \{\emptyset\}$

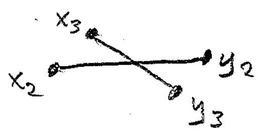
$K^{110110}(\Delta) =$



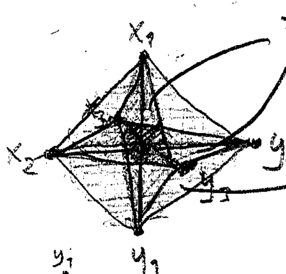
$K^{101101}(\Delta) =$



$K^{011011}(\Delta) =$



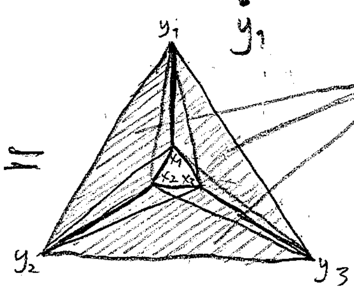
$K^{111111}(\Delta) =$



3 filled tetrahedra

no faces of the octahedron are present

$\cong$



3 filled tetrahedra