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2. Compute the homology groups for the simplicial complex on $\{a,b,c,d,e\}$ whose maximal faces are $\{a,b,c\},\{a,c,d\},\{a,d,e\},\{b,e\}$.

Denote the above defined simplicial complex as Δ (drawing on next page).

$F_{-1}(\Delta)=\{\emptyset\}$ $\qquad C_{-1}(\Delta)=\mathbb{F}^{\emptyset}=\text{span}(e_\emptyset)$

$F_0(\Delta)=\{a,b,c,d,e\}$ $\qquad C_0(\Delta)=\mathbb{F}^5=\text{span}(e_a,e_b,e_c,e_d,e_e)$

$F_1(\Delta)=\{ab,ac,ad,ae,bc,cd,de,be\}$ $\qquad C_1(\Delta)=\mathbb{F}^8\overset{\text{span}}{=}(e_{ab},e_{ac},e_{ad},e_{ae},e_{bc},e_{cd},e_{de},e_{be})$

$F_2(\Delta)=\{abc,acd,ade\}$ $\qquad C_2(\Delta)=\mathbb{F}^3=\text{span}(e_{abc},e_{acd},e_{ade})$

We have the chain complex

$$0\xrightarrow{\partial_3}C_2\xrightarrow{\partial_2}C_1\xrightarrow{\partial_1}C_0\xrightarrow{\partial_0}C_{-1}\xrightarrow{\partial_{-1}}0$$

Since ∂_{-1} maps C_{-1} to 0, $\ker(\partial_{-1})=\text{span}(e_\emptyset)$. And $\text{im}(\partial_0)=\text{span}(e_\emptyset)$ since

$$\partial_0(e_a)=\partial_0(e_b)=\partial_0(e_c)=\partial_0(e_d)=\partial_0(e_e)=e_\emptyset$$

Then

$$\widetilde{H}_{-1}(\Delta)=\ker(\partial_{-1})/\text{im}(\partial_0)=0$$

To compute $\widetilde{H}_0$, note that

$$\begin{aligned}\text{im}(\partial_1)&=\text{span}(e_a-e_b,e_a-e_c,e_a-e_d,e_a-e_e,e_b-e_c,e_c-e_d,e_d-e_e,e_b-e_e)\\&=\text{span}(e_a-e_b,e_b-e_c,e_c-e_d,e_d-e_e)\qquad\text{and}\\\ker(\partial_0)&=\{f=\alpha e_a+\beta e_b+\gamma e_c+\delta e_d+\epsilon e_e\mid\partial_0(f)=0)\end{aligned}$$

But $\partial_0(f)=(\alpha+\beta+\gamma+\delta+\epsilon)e_\emptyset=0$ implies that $\alpha+\beta+\gamma+\delta+\epsilon=0$. Therefore

$$\begin{aligned}\ker(\partial_0)&=\text{span}(e_a-e_b,e_a-e_c,e_a-e_d,e_a-e_e,e_b-e_c,e_b-e_d,e_b-e_e,e_c-e_d,e_c-e_e,e_d-e_e)\\&=\text{span}(e_a-e_b,e_b-e_c,e_c-e_d,e_d-e_e)\end{aligned}$$

Then

$$\widetilde{H}_0(\Delta)=\ker(\partial_0)/\text{im}(\partial_1)=0$$

To compute $\widetilde{H}_1$, notice that

$$\begin{aligned}\text{im}(\partial_2)&=\text{span}(-e_{bc}+e_{ac}-e_{ab},-e_{cd}+e_{ad}-e_{ac},-e_{de}+e_{ae}-e_{ad})\\\ker(\partial_1)&=\{f=k_1e_{ab}+k_2e_{ac}+k_3e_{ad}+k_4e_{ae}+k_5e_{bc}+k_6e_{cd}+k_7e_{de}+k_8e_{be}\mid\partial_1(f)=0\}\end{aligned}$$

But

$$\begin{aligned}\partial_1(f)&=k_1(e_a-e_b)+k_2(e_a-e_c)+k_3(e_a-e_d)+k_4(e_a-e_e)\\&\qquad+k_5(e_b-e_c)+k_6(e_c-e_d)+k_7(e_d-e_e)+k_8(e_b-e_e)\\&=(k_1+k_2+k_3+k_4)e_a+(-k_1+k_5+k_8)e_b+(-k_2-k_5+k_6)e_c+(-k_3-k_6+k_7)e_d+(-k_4-k_7-k_8)e_e\\&=0\end{aligned}$$

implies that $k_1=k_5+k_8,\quad k_2=-k_5+k_6,\quad k_3=-k_6+k_7,\quad k_4=-k_7-k_8$ (these were computed by row reducing the k_i coefficient matrix on a calculator). Then elements of the kernel of ∂_1 are of the form

$$\begin{aligned} f &= (k_5+k_8)e_{ab} + (-k_5+k_6)e_{ac} + (-k_6+k_7)e_{ad} + (-k_7-k_8)e_{ae} + k_5 e_{bc} + k_6 e_{cd} + k_7 e_{de} + k_8 e_{be} \\ &= k_5(e_{ab} - e_{ac} + e_{bc}) + k_6(e_{ac} - e_{ad} + e_{cd}) + k_7(e_{ad} - e_{ae} + e_{de}) + k_8(e_{ab} - e_{ae} + e_{be}) \end{aligned}$$

Therefore $\ker \partial_1 = \operatorname{span}((e_{ab} - e_{ac} + e_{bc}), (e_{ac} - e_{ad} + e_{cd}), (e_{ad} - e_{ae} + e_{de}), (e_{ab} - e_{ae} + e_{be}))$ and

$$\widetilde{H}_1(\Delta) = \ker(\partial_1)/\operatorname{im}(\partial_2) = \operatorname{span}(e_{ab} - e_{ae} + e_{be}) = \mathbb{F}^1$$

To compute $\widetilde{H}_2$, notice that $\operatorname{im}(\partial_3) = 0$, and $\ker(\partial_2) = \{f = k_1 e_{abc} + k_2 e_{acd} + k_3 e_{ade} \mid \partial_2(f) = 0\}$. But

$$\begin{aligned} \partial_2(f) &= k_1(-e_{bc} + e_{ac} - e_{ab}) + k_2(-e_{cd} + e_{ad} - e_{ac}) + k_3(-e_{de} + e_{ae} - e_{ad}) \\ &= -k_1 e_{ab} + (k_1 - k_2)e_{ac} + (k_2 - k_3)e_{ad} + k_3 e_{ae} - k_1 e_{bc} - k_2 e_{cd} - k_3 e_{de} \\ &= 0 \end{aligned}$$

implies that $k_1 = k_2 = k_3 = 0$, and therefore $\ker(\partial_2) = 0$. Then

$$\widetilde{H}_2(\Delta) = \ker(\partial_2)/\operatorname{im}(\partial_2) = 0/0 = 0$$

To summarize,

$$\begin{aligned} \widetilde{H}_{-1}(\Delta) &= 0 \\ \widetilde{H}_0(\Delta) &= 0 \\ \widetilde{H}_1(\Delta) &= \operatorname{span}(e_{ab} - e_{ae} + e_{be}) = \mathbb{F}^1 \\ \widetilde{H}_2(\Delta) &= 0 \end{aligned}$$