6(a) This is a special case of a more general fact, that if I is a squarefree monomial ideal of the ring R, then $\dim R/I$ is the maximal number of distinct variables of R whose product is not contained in I.

The Krull dimension of a ring S equals $1+\deg P(S;x)$ where P is the Hilbert polynomial. By the Stanley-Reisner machinery, we have

$$\deg P(S;d) = \sum_{F\in\Delta} \binom{d-1}{|F|-1}$$

where $I = I_\Delta$. The degree of this is the maximal degree of any term, that is $|F|-1$ where F is a largest face of Δ. But by definition the faces of Δ are the sets of variables whose product is not in I_Δ, and this is the claim.

(b) This is a question fit for Macaulay 2. The interaction runs as follows.

```
i1 : R = QQ[a,b,c,d,e,f,g,h,i];

i2 : M = coker matrix{{a*b*c,d*e*f,g*h*i,a*d*g,b*e*h,c*f*i,
a*e*i,b*f*g,c*d*h,a*f*h,b*d*i,c*e*g}};

i16 : dim M

o16 = 4

i11 : C = resolution M

       1       12       66       108       72       20       1
o11 = R  <-- R    <-- R    <-- R     <-- R    <-- R    <-- R  <-- 0

      0       1        2        3         4        5        6      7

o11 : ChainComplex

i18 : betti C

             0  1  2   3  4  5 6
o18 = total: 1 12 66 108 72 20 1
          0: 1  .  .   .  .  . .
          1: .  .  .   .  .  . .
          2: . 12  .   .  .  . .
          3: .  . 54  72 36  9 1
          4: .  . 12  36 36 11 .
```

```
o18 : BettiTally
```

In particular the maximum number of points in $\mathbb{F}_3^2$ with no SET is 4. It'd be easy to obtain the maps in the free resolution as well, but they're specified by fairly big matrices and I don't think it's enlightening to just plop the matrix down here. We know essentially what they must look like, anyway, by our monomial ideal machinery.

(c) Okay, let's see how far we Macaulay 2 can push this in a reasonable amount of time.

First, here's the slightly hackish but general enough way I've specified the $n = 3$ case to Macaulay; the approach generalises clearly.

```
i78 : R=QQ[x_((0,0,0))..x_((2,2,2))]

o78 = R

o78 : PolynomialRing

i100 : L = {{}}; for k from 1 to 3 do L = flatten for i from 0 to 2
list L/(t->append(t,i))

i106 : M = coker matrix {flatten for a in L list for b in L when a != b
list x_(toSequence a)*x_(toSequence b)*x_(toSequence (a+b)/(i->(6-i)%3))};
```

and then we can ask for dimensions and resolutions and Betti numbers. For $n = 3$, we quickly get $\dim M = 9$. But after that, we find that asking for a resolution for $n = 3$ or the dimension for $n = 4$ is quite a taxing thing for Macaulay 2 to do[1], and I cut the computations off after several hours. And I haven't thought of anything cleverer to do, so that's where it sits.

[1] I happen to know from SET lore that the dimension for $n = 4$ is 20.