6(a) One free resolution of $R/(x^m)$ is given as

$$\cdots \xrightarrow{x^m} R \xrightarrow{x^{n-m}} R \xrightarrow{x^m} R \xrightarrow{x^{n-m}} R \to R/(x^m) \to 0$$

where all the labelled arrows are multiplication. Exactness follows from that the image of multiplication by x^k in R is (x^k) and its kernel is (x^{n-k}), and that $R/(x^m)$ is isomorphic to (x^{n-m}) as an R-module. (This is routinely computed; each kernel one encounters in the computation has form (x^k), which has a single generator.)

(b) We'll show that every injection of free R-modules has free cokernel. Then given an R-module M with finite free resolution

$$0 \to F^n \to F^{n-1} \to \cdots \to F^1 \to F^0 \to M \to 0,$$

as long as $n > 0$ the left-hand end $0 \to F^n \xrightarrow{\phi} F^{n-1} \to \cdots$ of this sequence may be replaced with $0 \to (\operatorname{coker}\phi) \to \cdots$ while keeping the sequence a free resolution, and performing enough such replacements we find that M is isomorphic to a free R-module.

So let $\phi : G \hookrightarrow F$ be an injection of free R-modules. Now, ϕ restricts to an injection $x^{n-1}G \hookrightarrow x^{n-1}F$ of k-vector spaces, whose cokernel is a k-vector space as well. Let $\{x^{n-1}h_i : i \in N\}$ be representatives in $x^{n-1}G$ of a basis for this cokernel.

We then claim that $G/\phi(F)$ is freely generated as an R-module by the classes $\overline{h_i}$. By induction on n we may assume that the classes $\overline{h_i}$ freely generate $G/(x^{n-1}G + \phi(F))$.

Given any $g \in G$, there's some $R/(x^{n-1})$-linear combination of the h_i in $G/x^{n-1}G$, call it $\tilde{g}'$, such that $\overline{\tilde{g}'} = \overline{g}$ in $G/(x^{n-1}G + \phi(F))$. We lift $\tilde{g}'$ to $g' \in G$ by replacing the $R/(x^{n-1})$ coefficients with arbitrary lifts to $R/(x^n)$. Then $\overline{g' - g}$ in $G/\phi(F)$ is a multiple of x^{n-1}. So by choice of the h_i we have a $x^{n-1}R$-linear combination g'' of the h_i with $\overline{g''} = \overline{g' - g}$ in $G/\phi(F)$, and then $\overline{g} = \overline{g' - g''}$, showing that the h_i generate $G/\phi(F)$.

For freeness we do similarly. If we had $\sum_i a_i h_i \in \phi(F)$ for some $a_i \in R$, then inductively $a_i \cong 0 \bmod x^{n-1}$ for each i. But then $\sum_i a_i h_i$ is a k-linear combination of the $x^{n-1} h_i$, and so by choice of the h_i we must have that $a_i = 0$ for each i.