5. For all integers $n \geq 1$, let $R_n = \mathbb{F}[x_1, \dots, x_n]$ and define $R_\infty = \mathbb{F}[x_1, x_2, x_3, \dots]$. Let $I_n = \langle x_1, \dots, x_n \rangle$, the ideal generated in R_n.

Now, for $m \geq 1$ (we only use the case $m = 1$ anyway) consider the shift function,

$$f^{(m)}: \{x_1, x_2, x_3, \dots\} \to \{x_1, x_2, x_3, \dots\}$$
$$x_i \to x_{i+m}$$

This function extends to a monomorphism of rings $F^{(m)}: R_\infty \to R_\infty$, where $F^{(m)}(1) = 1$ necessarily.

Suppose we have a $l \times k$ matrix M with entries in R_∞ and define a new $l \times k$ matrix $M^{(m)}$, given by $M_{ij}^{(m)} = F^{(m)}(M_{ij})$ for all i and j. If we consider a vector $v \in R_\infty^k$, then $Mv = 0$ iff $M^{(m)} v^{(m)} = 0$ and this is true for all $m \geq 1$.

We will find a finite free resolution for our module inductively. For all $n \geq 1$, define the matrices

$$M_{\binom{n}{1}} = \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_n \end{pmatrix} \qquad (1.a)$$

and

$$M_{\binom{n}{n}} = \begin{pmatrix} x_n \\ -x_{n-1} \\ x_{n-2} \\ \vdots \\ (-1)^n x_2 \\ (-1)^{n-1} x_1 \end{pmatrix} \qquad (1.b)$$

For $n > 1$, suppose we know a resolution for $R_{n-1}/I_{n-1}$of the form,

$$0 \overset{0}{\longrightarrow} R_{n-1}^{\binom{n-1}{n-1}} \overset{M_{\binom{n-1}{n-1}}}{\longrightarrow} R_{n-1}^{\binom{n-1}{n-2}} \overset{M_{\binom{n-1}{n-2}}}{\longrightarrow} \dots \overset{M_{\binom{n-1}{2}}}{\longrightarrow} R_{n-1}^{\binom{n-1}{1}} \overset{M_{\binom{n-1}{1}}}{\longrightarrow} R_{n-1}^{\binom{n-1}{0}} \overset{\epsilon_{n-1}}{\longrightarrow} R_{n-1}/I_{n-1} \overset{0}{\longrightarrow} 0$$

where, for all $1 \le m \le n-1$, the matrix $M_{\binom{n-1}{m}}$ with entries in R_{n-1} induces the homomorphism-denoted by φ_m^{n-1}-from $R_{n-1}^{\binom{n-1}{m}}$ to $R_{n-1}^{\binom{n-1}{m-1}}$ and ϵ_{n-1} is the canonical homomorphism from R_{n-1} to R_{n-1}/I_{n-1}. The map φ_m^{n-1} is actually an epimorphism to the kernel of φ_{m-1}^{n-1} whenever $m > 1$. Also, we know-just by looking at the matrices-that $\ker(\varphi_{n-1}^{n-1}) = \{0\}$ and that $\mathrm{Im}(\varphi_1^{n-1}) = I_{n-1}$, the Kernel of the surjective map ϵ_{n-1}. We already know a resolution of this form for R_3/I_3 (obtained in class), but we will later compute a bigger case from it. We want to produce a similar resolution for R_n/I_n.

To start, for $1 < m < n$ define the matrix,

$$M_{\binom{n}{m}} = \left(\begin{array}{c|c} M^{(1)}_{\binom{n-1}{m-1}} & 0 \\ \hline (-1)^{m-1} x_1 I & M^{(1)}_{\binom{n-1}{m}} \end{array} \right) \qquad (2)$$

We claim that the sequence,

$$0 \overset{0}{\longrightarrow} R_n^{\binom{n}{n}} \overset{M_{\binom{n}{n}}}{\longrightarrow} R_n^{\binom{n}{n-1}} \overset{M_{\binom{n}{n-1}}}{\longrightarrow} \dots \overset{M_{\binom{n}{2}}}{\longrightarrow} R_n^{\binom{n}{1}} \overset{M_{\binom{n}{1}}}{\longrightarrow} R_{n-1}^{\binom{n}{0}} \overset{\epsilon_n}{\longrightarrow} R_n/I_n \overset{0}{\longrightarrow} 0$$

is a finite free resolution of R_n/I_n. Clearly, the map φ_1^n induced by $M_{\binom{n}{1}}$ has image equal to I_n (which is the Kernel of the surjective map ϵ_n) and $\operatorname{Ker}(\varphi_n^n) = \{0\}$. Now, for $1 < m < n-1$,

$$M_{\binom{n}{m}} M_{\binom{n}{m+1}} = \left[\begin{array}{c|c} M^{(1)}_{\binom{n-1}{m-1}} & 0 \\ \hline (-1)^{m-1} x_1 I & M^{(1)}_{\binom{n-1}{m}} \end{array}\right] \left[\begin{array}{c|c} M^{(1)}_{\binom{n-1}{m}} & 0 \\ \hline (-1)^{m} x_1 I & M^{(1)}_{\binom{n-1}{m+1}} \end{array}\right]$$

$$= \left(\begin{array}{c|c} M^{(1)}_{\binom{n-1}{m-1}} M^{(1)}_{\binom{n-1}{m}} & M^{(1)}_{\binom{n-1}{m-1}} 0 \\ \hline (-1)^{m-1} x_1 (I - I) & M^{(1)}_{\binom{n-1}{m}} M^{(1)}_{\binom{n-1}{m+1}} \end{array}\right)$$

$$= 0$$

by our induction hypothesis. Note that we have used the argument: $M_{\binom{n-1}{m-1}} M_{\binom{n-1}{m}} = 0$ implies $M^{(1)}_{\binom{n-1}{m-1}} M^{(1)}_{\binom{n-1}{m}} = 0$, a consequence of $F^{(1)}$ being a monomorphism of rings. So indeed, $\operatorname{Im}(\varphi_{m+1}^n) \subseteq \operatorname{Ker}(\varphi_m^n)$ for all such m. Also, if we let v_j be the j-th column of $M^{(1)}_{\binom{n-1}{2}}$, then using again the resolution for R_{n-1}/I_{n-1} we compute,

$$M_{\binom{n}{1}} M_{\binom{n}{2}} = (x_1\ x_2\ x_3 \ldots x_n) \left(\begin{array}{c|c} x_2 \ldots x_n & 0 \\ \hline -x_1 I & M^{(1)}_{\binom{n-1}{2}} \end{array}\right)$$

$$= \left(x_1 x_2 - x_2 x_1, x_1 x_3 - x_3 x_1, \ldots, x_1 x_n - x_n x_1, M^{(1)}_{\binom{n-1}{1}} v_1, \ldots, M^{(1)}_{\binom{n-1}{1}} v_{n-1}\right)$$

$$= 0$$

so $\text{Im}(\varphi_2^n) \subseteq \text{Ker}(\varphi_1^n)$. To study the remaining case, let u_i be the i-th row of $M^{(1)}_{\binom{n-1}{n-2}}$. We have,

$$M_{\binom{n}{n-1}} M_{\binom{n}{n}} = \left(\begin{array}{c|c} M^{(1)}_{\binom{n-1}{n-2}} & 0 \\ \hline (-1)^n x_1 I & \begin{matrix} x_n \\ \vdots \\ (-1)^n x_2 \end{matrix} \end{array} \right) \begin{pmatrix} x_n \\ -x_{n-1} \\ x_{n-2} \\ \vdots \\ (-1)^n x_2 \\ (-1)^{n-1} x_1 \end{pmatrix}$$

$$= \begin{pmatrix} -u_1 M^{(1)}_{\binom{n-1}{n-1}} \\ -u_2 M^{(1)}_{\binom{n-1}{n-1}} \\ \vdots \\ (-1)^{n-1}(-x_1 x_n + x_n x_1) \\ \vdots \\ x_1 x_2 - x_2 x_1 \end{pmatrix} = 0$$

where we have used once again the knowledge of the previous resolution for R_{n-1}/I_{n-1}. That is, $\text{Im}(\varphi_n^n) \subseteq \text{Ker}(\varphi_{n-1}^n)$.

The homomorphism properties of the maps φ_m for all $1 \leq m \leq n$ are immediate consequences of how they are obtained. Therefore, we need only justify why there's no strict containment in $\text{Im}(\varphi_{m+1}^n) \subseteq \text{Ker}(\varphi_m^n)$ for all $1 \leq m < n$ and we will be done. That $\text{Im}(\varphi_2^n) = \text{Ker}(\varphi_1^n)$ is a direct consequence of Schreyer's Theorem: The columns of matrix $M_{\binom{n}{2}}$ are simply the direct result of computing all the $S(x_i, x_j)$ with $i < j$ for the set $\{x_1, x_2, \dots, x_n\}$, which is a Gröbner basis for I_n, so they generate the module of sysygies of I_n and this is exactly $\text{Ker}(\varphi_1^n)$. Now, for $1 < m < n-1$, suppose we have a vector $v \in R_n^{\binom{n}{m}}$ such that $M_{\binom{n}{m}} v = 0$. Writing $v = \binom{v_1}{v_2}$ appropriately, this is equivalent to $M^{(1)}_{\binom{n-1}{m-1}} v_1 = 0$ and $(-1)^{m-1} x_1 v_1 + M^{(1)}_{\binom{n-1}{m}} v_2 = 0$. We may write the first of these equations as

$\sum_{i=0}^{k} x_1^i M^{(1)}_{\binom{n-1}{m-1}} v^{(1)}_{1i} = 0$ for some k, where $\sum_{i=0}^{k} x_1^i v^{(1)}_{1i} = v_1$ and $v^{(1)}_{1i}$ has entries in $F^{(1)}(R_\infty)$ for each i. More precisely, $v^{(1)}_{1i}$ is the vector with entries in the ring of polynomials in the variables $x_2, \dots, x_n$ associated via $F^{(1)}$ to a vector v_{1i}, whose entries are in R_{n-1}, as explained in the first part. But then we must have, for all i, $M^{(1)}_{\binom{n-1}{m-1}} v^{(1)}_{1i} = \left(M_{\binom{n-1}{m-1}} v_{1i}\right)^{(1)} = 0$, and this is true iff $M_{\binom{n-1}{m-1}} v_{1i} = 0$, so using the resolution for R_{n-1}/I_{n-1} we obtain $v_{1i} = M_{\binom{n-1}{m}} u_{1i}$ for some u_{1i} with entries in R_{n-1} and then $v^{(1)}_{1i} = M^{(1)}_{\binom{n-1}{m}} u^{(1)}_{1i}$. Finally, we can write $v_1 = \sum_{i=0}^{k} x_1^i\, M^{(1)}_{\binom{n-1}{m}} u^{(1)}_{1i} = M^{(1)}_{\binom{n-1}{m}} u_1$, u_1 having entries in R_n. Replacing this in the second equation we get $M^{(1)}_{\binom{n-1}{m}}((-1)^{m-1} x_1 u_1 + v_2) = 0$, and we use an analogous procedure to conclude that $(-1)^{m-1} x_1 u_1 + v_2 = M^{(1)}_{\binom{n-1}{m+1}} u_2$ for some vector u_2 with entries in R_n. Rearranging this equation, we get $v_2 = \left((-1)^m x_1 I \,\middle|\, M^{(1)}_{\binom{n-1}{m+1}}\right) \binom{u_1}{u_2} = \left((-1)^m x_1 I \,\middle|\, M^{(1)}_{\binom{n-1}{m+1}}\right) u$, where the vector u has just been defined, so (somehow amazingly) $v = M^{(1)}_{\binom{n}{m+1}} u$, $v \in \mathrm{Im}(\varphi^n_{m+1})$ and we have the equality $\mathrm{Im}(\varphi^n_{m+1}) = \mathrm{Ker}(\varphi^n_m)$ for all such m. The remaining case, where we want to prove $\mathrm{Im}(\varphi^n_n) = \mathrm{Ker}(\varphi^n_{n-1})$, can be established in very similar way, so our sequence is exact and it is a finite free resolution for R_n/I_n.

We know a finite free resolution like this for $n = 3$. It is given, from right to left by $0, \epsilon_3, (x_1\ x_2\ \ x_3), \begin{pmatrix} x_2 & x_3 & 0 \\ -x_1 & 0 & x_1 \\ 0 & -x_1 & -x_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix}$ and 0. Suppose we wanted to compute the resolution for $n = 4$. The first thing to do would be computing the matrices $M_{\binom{4}{1}}, M_{\binom{4}{2}}, M_{\binom{4}{3}}$ and $M_{\binom{4}{4}}$. Two of them we already know ($M_{\binom{4}{1}}$ and $M_{\binom{4}{4}}$) from Definition (2), to do the rest we need to find first $M^{(1)}_{\binom{3}{1}}, M^{(1)}_{\binom{3}{2}}, M^{(1)}_{\binom{3}{3}}$. We do this directly,

$$M^{(1)}_{\binom{3}{1}} = \begin{pmatrix} x_2 & x_3 & x_4 \end{pmatrix}$$

$$M^{(1)}_{\binom{3}{2}} = \begin{pmatrix} x_3 & x_4 & 0 \\ -x_2 & 0 & x_4 \\ 0 & -x_2 & -x_3 \end{pmatrix}$$

$$M^{(1)}_{\binom{3}{3}} = \begin{pmatrix} x_4 \\ -x_3 \\ x_2 \end{pmatrix}$$

But then, using Definition (2) (in red) before,

$$M_{\binom{4}{2}} = \begin{pmatrix} x_2 & x_3 & x_4 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & x_3 & x_4 & 0 \\ 0 & -x_1 & 0 & -x_2 & 0 & x_4 \\ 0 & 0 & -x_1 & 0 & -x_2 & -x_3 \end{pmatrix}$$

and also,

$$M_{\binom{4}{3}} = \begin{pmatrix} x_3 & x_4 & 0 & 0 \\ -x_2 & 0 & x_4 & 0 \\ 0 & -x_2 & -x_3 & 0 \\ x_1 & 0 & 0 & x_4 \\ 0 & x_1 & 0 & -x_3 \\ 0 & 0 & x_1 & x_2 \end{pmatrix}$$

This allows to complete the resolution for R_4/I_4. Using the pattern arising, one may compute much bigger cases easily. For example, $M_{\binom{6}{4}}$ has the form shown in the next page (empty spaces are 0's).

4	5	6												
-3			5	6										
	-3		-4		6									
		-3		-4	-5									
2						5	6							
	2					-2		6						
		2					-2	-5						
			2			3			6					
				2			3		-5					
					2			3	4					
-1										5	6			
	-1									-4		6		
		-1									-4	-5		
			-1							3			6	
				-1							3		-5	
					-1							3	4	
						-1				-2				6
							-1				-2			-5
								-1				-2		4
									-1				-2	-3