in $\mathbb{Q}[x,y]$ (or $\mathbb{Q}[x,y,z]$).

a) Determine whether $x^6 - x^5y$ is in $\langle \overset{f_1}{x^3-y}, \overset{f_2}{x^2y-y^2}, \overset{f_3}{xy^2-y^2}, \overset{f_4}{y^3-y^2} \rangle$.

① Check if $\{x^3-y, x^2y-y^2, xy^2-y^2, y^3-y^2\}$ is a G.b w.r.t. lex. or build it:

$$S(f_1,f_2) = yf_1 - xf_2 = -y^2 + xy^2 \equiv_{f_3} -y^2 + y^2 = 0$$

$$S(f_1,f_3) = y^2f_1 - x^2f_3 = x^2y^2 - y^3 \equiv_{f_2} y^3 - y^3 = 0$$

$$S(f_1,f_4) = y^3f_1 - x^3f_4 = x^3y^2 - y^4 \equiv_{f_1} y^3 - y^4 \equiv_{f_4} y^3 - y^3 = 0$$

$$S(f_2,f_3) = yf_2 - xf_3 = xy^2 - y^3 \equiv_{f_3} y^2 - y^3 \equiv_{f_4} 0$$

$$S(f_2,f_4) = y^2f_2 - x^2f_4 = x^2y^2 - y^4 \equiv_{f_2} y^3 - y^4 \equiv_{f_4} y^3 - y^3 = 0$$

$$S(f_3,f_4) = yf_3 - xf_4 = xy^2 - y^3 \equiv_{f_3} y^2 - y^3 \equiv_{f_4} 0$$

$\Rightarrow \{f_1, f_2, f_3, f_4\}$ is a G.b. w.r.t. lex.

② Determine whether $x^6 - x^5y \in I$: $g_0 = x^6 - x^5y$

$$g_1 = g_0 - x^3f_1 = x^3y - x^5y$$

$$g_2 = g_1 + x^2yf_1 = x^3y - x^2y^2$$

$$g_3 = g_2 - yf_1 = y^2 - x^2y^2$$

$$g_4 = g_3 + yf_2 = y^2 - y^3$$

$$g_5 = g_4 + f_4 = 0$$

$\Rightarrow x^6 - x^5y \in I$ ■

b) Determine whether $\langle \overset{f_1}{\underline{x^3}-yz}, \overset{f_2}{\underline{yz}+y} \rangle = \langle \overset{g_1}{\underline{x^3z}+x^3}, \overset{g_2}{\underline{x^3}+y} \rangle$.

① Determine wether $\{f_1, f_2\}$ and $\{g_1, g_2\}$ are a Gröbner basis:

$$S(f_1,f_2) = yzf_1 - x^3f_2 = -\underline{x^3y} - y^2z^2 \equiv_{f_1} -y^2z - \underline{y^2z^2} \equiv_{f_2} -y^2z + y^2z = 0$$

$$S(g_1, g_2) = g_1 - zg_2 = \underline{x^3} - zy \equiv_{g_2} -y - \underline{zy} \Rightarrow \text{take } g_3 = \underline{zy} + y$$

$$S(g_1, g_3) = yg_1 - x^3g_3 = x^3y - x^3y = 0$$

$$S(g_2, g_3) = zyg_2 - x^3g_3 = zy^2 - \underline{x^3y} \equiv_{g_2} \underline{zy^2} + y^2 \equiv_{g_3} -y^2 + y^2 = 0$$

$\Rightarrow \{f_1, f_2\}$ is a G.b for $\langle f_1, f_2\rangle$ and $\{g_1, g_2, g_3\}$ is a G.b for $\langle g_1, g_2\rangle$

② Construct/check if the G.b are reduced: all $in(f_i)$ are monic ✓

- since $in(f_2) = yz$ divides a monomial in f_1 we replace f_1 by its remainder dividing by f_2 $\Rightarrow$ $r = f_1 + f_2 = x^3 + y$

  $\Rightarrow \{x^3+y, yz+y\}$ is the reduced G.b for $\langle f_1, f_2\rangle$.

- Since $in(g_2)$ divides monomials in g_1, we replace g_1 by the remainder dividing by $\{g_2, g_3\}$:

$$g_1 = zg_2 + g_2 - g_3 + 0 \Rightarrow r = 0 \Rightarrow \underset{\{x^3+y,\, yz+y\}}{\overset{}{\{g_2, g_3\}}} \text{ is the reduced}$$

G.b. for $\langle g_1, g_2\rangle$.

③ Compare the reduced G.b.: Since the reduced G.b for $\langle f_1, f_2\rangle$ and $\langle g_1, g_2\rangle$ w.r.t lex are equal then the two ideals are equal.

(c) Solve the system of equations $\underset{f_1}{x^2-yz=3}$, $\underset{f_2}{y^2-xz=4}$, $\underset{f_3}{z^2-xy=5}$.

① Compute a G.b. for $\langle \underline{x^2}-yz-3,\ \underline{xz}-y^2+4,\ \underline{xy}-z^2+5 \rangle$:

$$S(f_1,f_2) = zf_1 - xf_2 = -yz^2 - 3z + \underline{xy^2} - 4x \equiv_{f_3} -\cancel{yz^2} - 3z + \cancel{z^2 y} - 5y - \underline{4x}$$

$$\Rightarrow \boxed{f_4 = \underline{4x} + 5y + 3z}$$

$$S(f_1,f_3) = yf_1 - xf_3 = \underline{xz^2} - 5x - y^2z - 3y \equiv_{f_2} \cancel{y^2z} - 4z - \underline{5x} - \cancel{y^2z} - 3y$$

$$\equiv_{f_4} -3y - 4z + \frac{25}{4}y + \frac{15}{4}z = \frac{13}{4}\underline{y} - \frac{1}{4}z$$

$$\Rightarrow \boxed{f_5 = \underline{13y} - z}$$

$$S(f_2,f_3) = yf_2 - zf_3 = \underline{-y^3} + 4y + z^3 - 5z \equiv_{f_5} -\frac{1}{13^3}z^3 + \underline{4y} + z^3 - 5z$$

$$\equiv_{f_5} \frac{13^3-1}{13^3}z^3 + \frac{4}{13}z - 5z = \frac{13^3-1}{13^3}z^3 + \frac{4-5\cdot 13}{13}z \Rightarrow f_6 = \underline{(13^3-1)\,z^3} + 13^2(4-5\cdot 13)z$$

$$S(f_1,f_4) = f_1 - \frac{1}{4}xf_4 = -yz - 3 - \frac{5}{4}\underline{xy} - \frac{3}{4}xz \equiv_{f_3} -yz - 3 - \frac{5}{4}z^2 + \frac{25}{4} - \underline{\frac{3}{4}xz}$$

$$\equiv_{f_2} -yz + \frac{13}{4} - \frac{5}{4}z^2 - \underline{\frac{3}{4}y^2} + 3 \equiv_{f_5} -\underline{yz} + \frac{25}{4} - \frac{5}{4}z^2 - \frac{3}{4\cdot 13^2}z^2$$

$$\equiv_{f_5} -\frac{1}{13}z^2 + \frac{25}{4} - \frac{5}{4}z^2 - \frac{3}{4\cdot 13^2}z^2 = \underline{-\frac{225}{169}z^2} + \frac{25}{4} \Rightarrow \boxed{f_6 = 36z^2 - 169}$$

$(f_6 \equiv 0 \pmod{36z^2-169})$)

$$S(f_2,f_4) = f_2 - \frac{1}{4}zf_4 = \underline{-y^2} + 4 - \frac{5}{4}yz - \frac{3}{4}z^2 \equiv_{f_5} -\frac{1}{13^2}z^2 + 4 - \frac{5}{4}\underline{yz} - \frac{3}{4}z^2$$

$$\equiv_{f_5} -\frac{1}{13^2}z^2 + 4 - \frac{5}{4\cdot 13}z^2 - \frac{3}{4}z^2 \equiv_{f_6} -\frac{1}{4\cdot 9} + 4 - \frac{5\cdot 13}{4^2\cdot 9} - \frac{13^2}{4^2\cdot 3} = 0$$

$$S(f_3,f_4) = f_3 - \frac{1}{4} y f_4 = \cancel{xy} - z^2 + 5 - \cancel{xy} - \frac{5}{4} y^2 - \frac{3}{4} yz$$

$$\equiv_{f_5} -z^2 + 5 - \frac{5}{4\cdot 13^2} z^2 - \frac{3}{4} yz \equiv_{f_5} -z^2 + 5 - \frac{5}{4\cdot 13^2} z^2 - \frac{3}{4\cdot 13} z^2$$

$$\equiv_{f_6} -\frac{13^2}{4\cdot 9} + 5 - \frac{5}{4^2\cdot 9} - \frac{13\cdot 3}{4^2\cdot 9} = 0$$

$$S(f_1,f_5) = y f_1 - \frac{1}{13} x^2 f_5 = -y^2 z - 3y + \frac{1}{13} x^2 z \equiv_{f_1} -y^2 z - 3y + \frac{1}{13} y z^2 + \frac{3}{13} z$$

$$\equiv_{f_5} -\frac{1}{13^2} z^3 - 3y + \frac{1}{13} y z^2 + \frac{3}{13} z \equiv_{f_6} -\frac{1}{13^2} z^3 - 3y + \frac{13}{4\cdot 9} y + \frac{3}{13} z$$

$$\equiv_{f_5} -\frac{1}{13^2} z^3 - \cancel{\frac{3}{13} z} + \frac{1}{4\cdot 9} z + \cancel{\frac{3}{13} z} \equiv_{f_6} -\frac{1}{4\cdot 9} z + \frac{1}{4\cdot 9} z = 0$$

$$S(f_2,f_5) = y f_2 - \frac{xz}{13} f_5 = -y^3 + 4y + \frac{xz^2}{13} \equiv_{f_2} -y^3 + 4y + \frac{y^2 z}{13} - \frac{4}{13} z$$

$$\equiv_{f_5} -\frac{1}{13^3} z^3 + 4y + \frac{y^2 z}{13} - \frac{4}{13} z \equiv_{f_5} -\cancel{\frac{1}{13^3} z^3} + 4y + \cancel{\frac{1}{13^3} z^3} - \frac{4}{13} z$$

$$\equiv_{f_5} \frac{4}{13} z - \frac{4}{13} z = 0$$

$$S(f_3,f_5) = f_3 - \frac{1}{13} x f_5 = -z^2 + 5 + \frac{1}{13} xz \equiv_{f_2} -z^2 + 5 + \frac{1}{13} y^2 - \frac{4}{13} \equiv_{f_5} -z^2 + 5 + \frac{1}{13^3} z^2 - \frac{4}{13}$$

$$\equiv_{f_6} -\frac{13^2}{4\cdot 9} + 5 + \frac{1}{13\cdot 4\cdot 9} - \frac{4}{13} = 0$$

$$S(f_4,f_5) = \frac{1}{4} y f_4 - \frac{1}{13} x f_5 = \frac{5}{4} y^2 + \frac{3}{4} yz + \frac{1}{13} xz \equiv_{f_2} \frac{5}{4} y^2 + \frac{3}{4} yz + \frac{1}{13} y^2 - \frac{4}{13}$$

$$\equiv_{f_5} \frac{5}{4\cdot 13^2} z^2 + \frac{3}{4} yz + \frac{1}{13^3} z^2 - \frac{4}{13} \equiv_{f_5} \frac{5}{4\cdot 13^2} z^2 + \frac{3}{4\cdot 13} z^2 + \frac{1}{13^3} z^2 - \frac{4}{13}$$

$$\equiv_{f_6} \frac{5}{4^2\cdot 9} + \frac{13\cdot 3}{4^2\cdot 9} + \frac{1}{13\cdot 4\cdot 9} - \frac{4}{13} = 0$$

$$S(f_1,f_6) = z^2 f_1 - \frac{1}{4\cdot 9}x^2 f_6 = -yz^3 - 3z^2 + \frac{13^2}{4\cdot 9}x^2 \equiv_{f_1} -yz^3 - 3z^2 + \frac{13^2}{4\cdot 9}yz + \frac{13^2}{4\cdot 3}$$

$$\equiv_{f_6} -\cancel{\frac{13^2}{4\cdot 9}yz} - 3z^2 + \cancel{\frac{13^2}{4\cdot 9}yz} + \frac{13^2}{4\cdot 3} \equiv_{f_6} -\frac{13^2}{4\cdot 3} + \frac{13^2}{4\cdot 3} = 0$$

$$S(f_2,f_6) = zf_2 - \frac{1}{4\cdot 9}x f_6 = -y^2z + 4z + \frac{13^2}{4\cdot 9}x \equiv_{f_4} -y^2z + 4z - \frac{13^2\cdot 5}{4^2\cdot 9}y - \frac{13^2\cdot 3}{4^2\cdot 9}z$$

$$\equiv_{f_5} -\frac{1}{13^2}z^3 + 4z - \frac{13^2\cdot 5}{4^2\cdot 9}y - \frac{13^2}{4^2\cdot 3}z \equiv_{f_5} -\frac{1}{13^2}z^3 + 4z - \frac{13\cdot 5}{4^2\cdot 9}z - \frac{13^2}{4^2\cdot 3}z$$

$$\equiv_{f_6} -\frac{1}{4\cdot 9}z + 4z - \frac{13\cdot 5}{4^2\cdot 9}z - \frac{13^2\cdot 3}{4^2\cdot 9}z = 0$$

$$S(f_3,f_6) = z^2 f_3 - \frac{1}{4\cdot 9}xy f_6 = -z^4 + 5z^2 + \frac{13^2}{4\cdot 9}xy \equiv_{f_3} -z^4 + 5z^2 + \frac{13^2}{4\cdot 9}z^2 - \frac{5\cdot 13^2}{4\cdot 9}$$

$$\equiv_{f_6} -\frac{13^4}{4^2\cdot 9^2} + 5z^2 + \frac{13^2}{4\cdot 9}z^2 - \frac{5\cdot 13^2}{4\cdot 9} \equiv_{f_6} -\frac{13^4}{4^2\cdot 9^2} + \frac{5\cdot 13^2}{4\cdot 9} + \frac{13^4}{4^2\cdot 9^2} - \frac{5\cdot 13^2}{4\cdot 9} = 0$$

$$S(f_4,f_6) = \frac{1}{4}z^2 f_4 - \frac{1}{4\cdot 9}x f_6 = \frac{5}{4}yz^2 + \frac{3}{4}z^3 + \frac{13^2}{4\cdot 9}x$$

$$\equiv_{f_1} \frac{5}{4}yz^2 + \frac{3}{4}z^3 - \frac{13^2\cdot 5}{4^2\cdot 9}y - \frac{13^2\cdot 3}{4^2\cdot 9}z \equiv_{f_6} \cancel{\frac{5\cdot 13^2}{4^2\cdot 9}y} + \frac{3}{4}z^3 - \cancel{\frac{13^2\cdot 5}{4^2\cdot 9}y} - \frac{13^2\cdot 3}{4^2\cdot 9}z$$

$$\equiv_{f_6} \frac{3\cdot 13^2}{4^2\cdot 9}z - \frac{13^2\cdot 3}{4^2\cdot 9}z = 0$$

$$S(f_5,f_6) = \frac{1}{13}z^2 f_5 - \frac{1}{4\cdot 9}y f_6 = -\frac{1}{13}z^3 + \frac{13^2}{4\cdot 9}y \equiv_{f_5} -\frac{1}{13}z^3 + \frac{13}{4\cdot 9}z \equiv_{f_6} -\frac{13}{4\cdot 9}z + \frac{13}{4\cdot 9}z = 0$$

$\boxed{\Rightarrow \{f_1, f_2, f_3, f_4, f_5, f_6\} \text{ is a G.b for } \langle f_1, f_2, f_3\rangle}$, actually we will consider the Gröbner Basis $G = \{f_4, f_5, f_6\}$ for this ideal (since the initial terms of these monomials divide those of f_1, f_2).

② Solve the system of equations:

a) $G \cap \mathbb{F}[z] = \{36z^2 - 169\}$, the solutions to this equation are

$z = \pm \frac{13}{6}$

b) $G \cap \mathbb{F}[y,z] = \{13y - z,\ 36z^2 - z\}$, to solve the system

$$\begin{aligned} 13y - z &= 0 \\ 36z^2 - z &= 0 \end{aligned}$$

we substitute $z = \pm \frac{13}{6}$ and solve the system.

Then the points $(\overset{y}{1/6}, \overset{z}{13/6})$ and $(\overset{y}{-1/6}, \overset{z}{-13/6})$ are the solutions to the system.

c) $G \cap \mathbb{F}[x,y,z] = \{4x + 5y + 3z,\ 13y - z,\ 36z^2 - 169\}$, and the solutions for the system given by these equations can be found by substituting the previous points into the first equation and finding x.

- $4x + \frac{5}{6} + \frac{39}{6} = 0 \Rightarrow x = -\frac{11}{6}$

- $4x - \frac{5}{6} - \frac{39}{6} = 0 \Rightarrow x = \frac{11}{6}$

♡ Therefore the solutions to the system are the points $P_1 = \left(-\frac{11}{6}, \frac{1}{6}, \frac{13}{6}\right)$ and $P_2 = \left(\frac{11}{6}, -1/6, -13/6\right)$.

Note: here we used the fact that $V(\{f_1, \dots, f_k\}) = V(\langle f_1, \dots, f_k \rangle)$ because we found $V(\{f_4, f_5, f_6\}) = V(\langle f_1, f_2, f_3 \rangle) = V(\{f_1, f_2, f_3\}) = \{P_1, P_2\}$.

d) Compute $\overset{I}{\langle x^3y-xy^2+1,\ x^2y^2-y^3-1\rangle} \cap \overset{J}{\langle x^2-y^2,\ x^3+y^3\rangle}$.

$$\langle t\rangle I + \langle 1-t\rangle J = \langle \overset{f_1}{t(x^3y-xy^2+1)},\ \overset{f_2}{t(x^2y^2-y^3-1)},\ \overset{f_3}{(1-t)(x^2-y^2)},\ \overset{f_4}{(1-t)x^3+y^3}\rangle$$

lets compute a G.b for this ideal wrt lex $t>x>y$.

$$S(f_1,f_2) = yf_1 - xf_2 = -\cancel{txy^3} + ty + \cancel{txy^3} + tx \Rightarrow \boxed{f_5 = tx+ty}$$

$$S(f_1,f_3) = f_1 + xyf_3 = -txy^2+t+x^3y-xy^3+txy^3 \equiv_{f_5} -txy^2+t+x^3y-xy^3-ty^4$$

$$\equiv_{f_5} ty^3+t+x^3y-xy^3-ty^4 \Rightarrow \boxed{f_6 = ty^4-ty^3-t-x^3y+xy^3}$$

$$S(f_1,f_4) = f_1 + yf_4 = -txy^2+t+x^3y+y^4-ty^4 \equiv_{f_5} ty^3+t+x^3y+y^4-ty^4$$

$$\equiv_{f_6} \cancel{ty^3}+\cancel{t}+\cancel{x^3y}+y^4-\cancel{ty^3}-\cancel{t}-\cancel{x^3y}+xy^3 = xy^3+y^4$$

$$\Rightarrow \boxed{f_7 = xy^3+y^4}$$

$$S(f_2,f_3) = f_2 + y^2f_3 = -ty^3-t+x^2y^2-y^4+ty^4 \equiv_{f_6} -\cancel{ty^3}-\cancel{t}+x^2y^2-y^4+\cancel{ty^3}+\cancel{t}+x^3y-xy^3$$

$$= x^3y+x^2y^2-xy^3-y^4 \Rightarrow \boxed{f_8 = x^3y+x^2y^2-xy^3-y^4}$$

$$S(f_2,f_4) = xf_2 + y^2f_4 = -txy^3-tx+x^3y^2+y^5-ty^5 \equiv_{f_5} ty^4-tx-ty^5+x^3y^2+y^5$$

$$\equiv_{f_5} ty^4+ty-ty^5+x^3y^2+y^5 \equiv_{f_6} xy^4+y^5 \equiv_{f_7} 0.$$

$$S(f_3,f_4) = xf_3 - f_4 = \cancel{x^3}-xy^2+txy^2-\cancel{x^3}-y^3+ty^3 \equiv_{f_5} -xy^2-\cancel{ty^3}-y^3+\cancel{ty^3}$$

$$\Rightarrow \boxed{f_7 = xy^2+y^3} \qquad (\text{old } f_7 = y(xy^2+y^3))$$

$$S(f_5,f_1) = f_1 - x^2yf_5 = -txy^2+t-tx^2y^2 \equiv_{f_2} -txy^2+\cancel{t}-ty^3-\cancel{t} \equiv_{f_7} 0$$

$$S(f_2,f_5) = f_2 - xy^2 f_5 = -ty^3 - t - \underline{txy^3} \equiv_{f_5} -ty^3 - t + \underline{ty^4} \equiv_{f_6} \cancel{-ty^3} \cancel{-t} \cancel{+ty^3} \cancel{+t} \underline{+x^3y} - xy^3$$

$$\equiv_{f_8} \underline{-x^2y^2} \cancel{+xy^3} + y^4 \cancel{-xy^3} \equiv_{f_7} \underline{xy^3} + y^4 \equiv_{f_7} 0$$

$$S(f_3,f_5) = f_3 + xf_5 = x^2 - y^2 + ty^2 + \underline{txy} \equiv_{f_5} \underline{x^2} - y^2 \cancel{+ty^2} \cancel{-ty^2}$$

$$\Rightarrow \boxed{f_3 = \underline{x^2} - y^2} \qquad (\text{old } f_3 = (1-t)(x^2-y^2))$$

$$S(f_4,f_5) = f_4 + x^2 f_5 = x^3 + y^3 - ty^3 + \underline{tx^2y} \equiv_{f_5} x^3 + y^3 - ty^3 - \underline{txy^2} \equiv_{f_5} \underline{x^3} + y^3$$

$$\equiv_{f_3} \underline{xy^2} + y^3 \equiv_{f_7} 0$$

$$S(f_1,f_6) = y^3 f_1 - x^3 f_6 = -txy^5 + ty^3 + \underline{tx^3y^3} + tx^3 + x^6y - x^4y^3$$

$$\equiv_{f_3} \cancel{-txy^5} + ty^3 \cancel{+txy^5} + \underline{tx^3} + x^6y - x^4y^3 \equiv_{f_5} ty^3 - tx^2y + x^6y - x^4y^3$$

$$\equiv_{f_5} ty^3 + txy^2 + x^6y - x^4y^3 \equiv_{f_5} \underline{x^6y} - x^4y^3 \equiv_{f_3} 0$$

$$S(f_2,f_6) = y^2 f_2 - x^2 f_6 = -ty^5 - ty^2 + \underline{tx^2y^3} + tx^2 + x^5y - x^3y^3 \equiv_{f_8} -ty^2 + \underline{tx^2} + x^5y - x^3y^3$$

$$\equiv_{f_3} \underline{x^5y} - x^3y^3 \equiv_{f_3} 0$$

$$S(f_3,f_6) = ty^4 f_3 - x^2 f_6 = -ty^6 + \underline{tx^2y^3} + tx^2 + x^5y - x^3y^3 \equiv_{f_3} -\underline{ty^6} + ty^5 + ty^2 + x^5y - x^3y^3$$

$$\equiv_{f_6} \cancel{-ty^5} \cancel{-ty^2} - x^3y^3 + xy^5 \cancel{+ty^5} \cancel{+ty^2} + x^5y - x^3y^3 = \underline{x^5y} - 2x^3y^3 + xy^5$$

$$\equiv_{f_3} -x^3y^3 + xy^5 \equiv_{f_3} 0$$

$$S(f_4,f_6) = y^4 f_4 + x^3 f_6 = x^3y^4 + y^7 - ty^7 - \underline{tx^3y^3} - tx^3 - x^6y + x^4y^3 \equiv_{f_3} x^3y^4 + y^7 - ty^7 - txy^5 - \underline{tx^3} - x^6y + x^4y^3$$

$$\equiv_{f_3} x^3y^4 + y^7 - ty^7 - \underline{txy^5} - txy^2 - x^6y + x^4y^3 \equiv_{f_5} x^3y^4 + y^7 - \underline{ty^7} + ty^6 + ty^3 - x^6y + x^4y^3$$

$$\equiv_{f_6} \cancel{x^3y^4} + y^7 \cancel{-ty^6} \cancel{-ty^3} \cancel{-x^3y^4} + xy^6 \cancel{+ty^6} \cancel{+ty^3} - x^6y + x^4y^3$$

$$= -\underline{x^6y} + x^4y^3 + xy^6 + y^7 \equiv_{f_3} \cancel{-x^4y^3} \cancel{+x^4y^3} + \underline{xy^6} + y^7 \equiv_{f_7} 0$$

$$S(f_5,f_6) = y^4 f_5 - x f_6 = ty^5 + txy^3 + tx + x^4y - x^2y^3 \equiv_{f_7} ty^5 - ty^4 + tx + x^4y - x^2y^3$$

$$\equiv_{f_5} ty^5 - ty^4 - ty + x^4y - x^2y^3 \equiv_{f_6} \cancel{ty^4} + ty \mp x^3y^2 - xy^4 - \cancel{ty^4} + tx + x^4y - x^2y^3$$

$$\equiv_{f_5} x^3y^2 - xy^4 + x^4y - x^2y^3 \equiv_{f_3} x^3y^2 - xy^4 \equiv_{f_3} 0$$

$$S(f_1,f_7) = y f_1 - tx^2 f_7 = -txy^3 + ty - tx^2y^3 \equiv_{f_3} -txy^3 + ty - ty^5 \equiv_{f_5} ty^4 + ty - ty^5$$

$$\equiv_{f_6} \cancel{ty^4} + \cancel{ty} - \cancel{ty^4} - \cancel{ty} - x^3y^2 + xy^4 \equiv_{f_3} 0$$

$$S(f_2,f_7) = f_2 - txf_7 = -ty^3 - t - txy^3 \equiv_{f_5} -ty^3 - t + ty^4$$

$$\equiv_{f_6} \cancel{-ty^3} \cancel{-t} + \cancel{ty^3} + \cancel{t} + x^3y - xy^3 \equiv_{f_3} 0$$

$$S(f_3,f_7) = y^2 f_3 - x f_7 = -y^4 - xy^3 \equiv_{f_7} 0$$

$$S(f_4,f_7) = y^2 f_4 + tx^2 f_7 = x^3y^2 + y^5 - ty^5 + tx^2y^3 \equiv_{f_3} x^3y^2 + y^5 \equiv_{f_3} xy^3 + y^5 \equiv_{f_7} 0$$

$$S(f_5,f_7) = y^2 f_5 - t f_7 = ty^3 - ty^3 = 0$$

$$S(f_6,f_7) = x f_6 - ty^2 f_7 = -txy^3 - tx - x^4y + x^2y^3 - ty^5 \equiv_{f_5} ty^4 + ty - x^4y + x^2y^3 - ty^5$$

$$\equiv_{f_6} \cancel{ty^4} + \cancel{ty} \cancel{- x^4y + x^2y^3} \cancel{- ty^4} \cancel{- ty} - x^3y^2 + xy^4 \equiv_{f_3} -x^3y^2 + xy^4 \equiv_{f_3} 0$$

$$S(f_1,f_8) = f_1 - t f_8 = -txy^2 + t - tx^2y^2 + txy^3 + ty^4 \equiv_{f_3} -txy^2 + t + txy^3$$

$$\equiv_{f_5} ty^3 + t - ty^4 \equiv 0$$

$$S(f_2,f_8) = x f_2 - ty f_8 = -txy^3 - tx - tx^2y^3 + txy^4 + ty^5 \equiv_{f_3} -txy^3 - tx + txy^4 \equiv 0$$

$$S(f_3,f_8) = xy f_3 - f_8 = \cancel{-xy^3} - x^2y^2 + \cancel{xy^3} + y^4 \equiv_{f_3} 0$$

$$S(f_4,f_8) = y f_4 + t f_8 = x^3y + y^4 - 2ty^4 + tx^2y^2 - txy^3 \equiv_{f_3} x^3y + y^4 - ty^4 - txy^3$$

$$\equiv_{f_5} x^3y + y^4 \equiv_{f_3} xy^3 + y^4 \equiv_{f_7} 0$$

$$S(f_5,f_8) = x^2y f_5 - t f_8 = \cancel{tx^2y^2} - \cancel{tx^2y^2} + txy^3 + ty^4 \equiv_{f_5} 0$$

$$S(f_6,f_8) = x^3 f_6 - ty^3 f_8 = -tx^3y^3 - tx^3 - x^6y + x^4y^3 - tx^2y^5 + txy^6 + ty^7$$

$$\equiv_{f_3} -txy^5 - txy^2 - x^6y + x^4y^3 - tx^2y^5 + txy^6 + ty^7 \equiv_{f_3} -txy^5 - txy^2 - x^6y + x^4y^3 + txy^6$$

$$\equiv_{f_5} ty^6 + ty^3 - x^6y + x^4y^3 - ty^7 \equiv_{f_6} \cancel{ty^6} + \cancel{ty^3} - \cancel{x^6y} + x^4y^3 - \cancel{ty^6} - \cancel{ty^3} - x^3y^4 + xy^6$$

$$\equiv_{f_3} -x^3y^4 + xy^6 \equiv_{f_3} 0$$

$$S(f_7,f_8) = x^2 f_7 - y f_8 = \cancel{x^2y^3} - \cancel{x^2y^3} + xy^4 + y^5 \equiv_{f_7} 0$$

$\Rightarrow$ $\boxed{\{f_1,\dots,f_8\} = G \text{ is a Gröbner basis for the desired ideal}}$

And thus $I \cap J = \langle G \cap \mathbb{F}[x,y] \rangle = \langle x^2 - y^3, xy^2 + y^3, x^3y + x^2y^2 - xy^3 - y^4 \rangle$.

(e) Compute the syzygies between the polynomials $\overset{f_1}{x^2}$, $\overset{f_2}{y^2}$, and $\overset{f_3}{xy + yz}$.

$S(f_1,f_2) = y^2 f_1 - x^2 f_2 = 0 \Rightarrow \begin{bmatrix} y^2 \\ -x \\ 0 \end{bmatrix}$ is a syzygy

$S(f_1,f_3) = y f_1 - x f_3 = -xyz = -z f_3 + yz^2 \Rightarrow \boxed{f_4 = yz^2}$ and $\boxed{f_4 = y f_1 + (z-x) f_3}$

$S(f_2,f_3) = x f_2 - y f_3 = -y^2 z = -z f_2 \Rightarrow \begin{bmatrix} 0 \\ x+z \\ -y \end{bmatrix}$ is a syzygy

$S(f_1,f_4) = yz^2 f_1 - x^2 f_4 = 0$

$S(f_2,f_4) = z^2 f_2 - y f_4 = 0$

$S(f_3,f_4) = z^2 f_3 - x f_4 = yz^3 = z f_4 \Rightarrow z^2 f_3 - (x+z) f_4 = 0$

Since we introduced f_4 we need to substitute f_4 in each of the relations found.

From $S(f_1,f_4)$ we get $yz^2 f_1 - x^2 y f_1 + x^2(x-z) f_3 = 0$ so we get the syzygy $\begin{bmatrix} yz^2 - x^2y \\ 0 \\ x^3 - x^2z \end{bmatrix}$.

from $S(f_2, f_4)$ we get $z^2 f_2 - y^2 f_1 + (xy - yz) f_3 = 0$ and thus the syzygy $\begin{bmatrix} -y^2 \\ z^2 \\ xy-yz \end{bmatrix}$.

From $S(f_3, f_4)$ we get $z^2 f_3 - y(x+z) f_1 + (x+z)(x-z) f_3 = 0$ and thus the syzygy $\begin{bmatrix} -xy-yz \\ 0 \\ x^2 \end{bmatrix}$

Therefore the syzygies among the given polynomials are generated by

$$\left\{ \begin{bmatrix} y^2 \\ -x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x+y \\ -y \end{bmatrix}, \begin{bmatrix} yz^2 - x^2 y \\ 0 \\ x^3 - x^2 z \end{bmatrix}, \begin{bmatrix} -y^2 \\ z^2 \\ xy-yz \end{bmatrix}, \begin{bmatrix} -xy-yz \\ 0 \\ x^2 \end{bmatrix} \right\}$$