1. (a) Let P be the partially ordered set of (monic) monomials in the variables $x_1, \ldots, x_n$, where $m_1 \leq m_2$ if and only if m_1 divides m_2. Give a combinatorial proof (without using Hilbert's basis theorem) of the fact that any subset of P has a finite number of minimal elements.

   Let us identify P with $\mathbf{Z}_{\geq 0}^n$, where each monomial is represented by its multideg vector. Then for monomials $m_1 = (a_1, \ldots, a_n)$ and $m_2 = (b_1, \ldots, b_n)$, we have $m_1 \leq m_2$ in our partial order whenever $a_i \leq b_i$ for all $i \in \{1, \ldots, n\}$. Now let $S \subset P$. Let $m = (a_1, \ldots, a_n)$ be any minimal element of S (if there are none, we're done). Now for each $i \in \{1, \ldots, n\}$ and $j \in \{0, \ldots, a_i - 1\}$, define the slice of S given by $T_{ij} := \{(b_1, \ldots, b_n) \in S \mid b_i = j\}$. Clearly, there are only finitely many such slices, and all of the minimal elements of S other than m must be in one of those slices (otherwise, they would be strictly larger than m). Furthermore, each slice has only $n-1$ "dimensions" (that is, it looks like a subset of $\mathbf{Z}_{\geq 0}^{n-1}$), simplifying the problem. We may now proceed by induction, since if each $(n-1)$-dimensional slice only has finitely many minimal elements, then there are finitely many minimal elements total. But in the one-dimensional case, there is clearly at most one minimal element, so by induction, we have what we wanted.

   (b) Use part (a) to show that every ideal of $\mathbf{F}[x_1, \ldots, x_n]$ has a finite Grobner basis, and hence is finitely generated.

   Let I be an ideal of $\mathbf{F}[x_1, \ldots, x_n]$. Then consider the set S of monomials in $\text{in}(I)$. By (a), S, under the above partial order, has a finite number of minimal elements, $\{m_1, \ldots, m_k\}$, which clearly generate $\text{in}(I)$ (since any monomial in $\text{in}(I)$ is either a multiple of an m_i or a minimal element itself, and hence among the m_i). Then since $m_i \in \text{in}(I)$ for each i, there are polynomials $\{g_1, \ldots, g_k\} \subset I$ such that $\text{in}(g_i) = m_i$ for each i. Thus, since the initial terms of the g_i generate the initial ideal of I, they form a (finite) Gröbner basis.