d. Suppose we have a monomial ordering $<_{\text{mon}}$ and let C be the set of vectors $\nu \in \mathbb{Z}^n$ such that $\nu = a - b$ for monomials $\mathbf{x}^b <_{\text{mon}} \mathbf{x}^a$.

Result 1. If $\nu_1, \nu_2 \in C$ and $p, q \in \mathbb{Q}_+$ then $p\nu_1 + q\nu_2 \in C$ whenever $p\nu_1 + q\nu_2 \in \mathbb{Z}^n$.

Proof. Consider four monomials $\mathbf{x}^b <_{\text{mon}} \mathbf{x}^a$ and $\mathbf{x}^d <_{\text{mon}} \mathbf{x}^c$ in $\mathbb{F}[x_1, \ldots, x_n]$. Suppose we have $p, q \in \mathbb{Q}_+$ such that $\nu = p(a-b) + q(c-d) \in \mathbb{Z}^n$. It is convenient to explicitly write p and q as fractions:

$$p = \frac{p_N}{p_D},\ q = \frac{q_N}{q_D} \text{ with } p_D, q_D, p_N, q_N \in \mathbb{Z}_+$$

We know $\left(\mathbf{x}^b\right)^{p_N q_D} <_{\text{mon}} (\mathbf{x}^a)^{p_N q_D}$ and $\left(\mathbf{x}^d\right)^{p_D q_N} <_{\text{mon}} (\mathbf{x}^c)^{p_D q_N}$. Defining

$$f = p_N q_D a + p_D q_N c$$

$$g = p_N q_D b + p_D q_N d$$

we have $\mathbf{x}^g <_{\text{mon}} \mathbf{x}^f$ so $f - g \in C$. It is easy to check that $-\mu \notin C$ whenever $\mu \in C$. Now, choose $e \in \mathbb{Z}^n_{\geq 0}$ so that $e + \nu \in \mathbb{Z}^n_{\geq 0}$, then $(\mathbf{x}^e)^{p_D q_D} = \mathbf{x}^{p_D q_D e}$ and $\left(\mathbf{x}^{e+\nu}\right)^{p_D q_D} = \mathbf{x}^{p_D q_D e + f - g}$. Because $f - g$ is in C and $<_{\text{mon}}$ is a total ordering on monomials, we have $\mathbf{x}^{p_D q_D e} <_{\text{mon}} \mathbf{x}^{p_D q_D e + f - g}$ so $(\mathbf{x}^e)^{p_D q_D} <_{\text{mon}} (\mathbf{x}^{e+\nu})^{p_D q_D}$. But then it must be true that $\mathbf{x}^e <_{\text{mon}} \mathbf{x}^{e+\nu}$ so $\nu \in C$. $\square$

Suppose we are given a finite number of vectors $\nu_1, \nu_2, \ldots, \nu_m \in C$ and consider the convex combination

$$\sum_{i=1}^{m} r_i \nu_i = 0 \tag{1}$$

with $r_1, \ldots, r_m \in \mathbb{R}_+$. Clearly $m \geq 2$. Let M be the $n \times m$ matrix with i-th column equal to ν_i and let $r = (r_1, \ldots, r_m)$. The null space of the linear map L_M induced by M is nontrivial because $L_M(r) = 0$ and has a basis β consisting of vectors in $\mathbb{Q}^m$. This may be checked by Gaussian elimination on M. Say $\beta = \{u_1, \ldots, u_l\}$ and write $r = c_1 u_1 + \cdots + c_l u_l$. As r lies in $\mathbb{R}^m_+$ we may find rationals $c_1^*, \ldots, c_l^*$ sufficiently close to $c_1, \ldots, c_l$ (respectively) such that $q = c_1^* u_1 + \cdots + c_l^* u_l$ has positive components, *i.e.* $q \in \mathbb{Q}^m_+$. We can choose these rational coefficients so that the components of q add up to 1. But then $L_M(q) = 0$ and we have the rational convex combination

$$\sum_{i=1}^{m} (q)_i \nu_i = 0$$

More conveniently, we have positive integers $n_1, \ldots, n_m$ such that

$$\sum_{i=1}^{m} n_i \nu_i = 0$$

Using Result 1 inductively we obtain a contradiction because clearly $0 \notin C$. Note Equation 1 holds for vectors $\nu_1, \ldots, \nu_m \in C$ iff 0 lies in the convex hull of C, denoted by $\mathrm{ch}(C)$. Thus $0 \notin \mathrm{ch}(C)$.

One separation theorem in convex analysis shows there exists an hyperplane $H^{(1)}$ of $\mathbb{R}^n$ separating (not necessarily strictly) 0 and $\mathrm{ch}(C)$, *i.e.* there exists a vector $v_1 \in \mathbb{R}^n \setminus \{0\}$ such that $v_1 \cdot x \geq 0$ for all $x \in \mathrm{ch}(C)$. Actually v_1 has nonnegative components because C contains the canonical basis of $\mathbb{R}^n$. Moreover, if μ is a vector in $\mathbb{Z}^n \setminus \{0\}$ such that $v_1 \cdot \mu > 0$ then μ lies in C because either $\mu \in C$ or $-\mu \in C$ holds and in the later case we would have $v_1 \cdot \mu \leq 0$. We may simply define $H^{(1)} = v_1^{\perp}$, the orthogonal complement of v_1.

Now, define the set $C^{(1)} = H^{(1)} \cap C$. If $C^{(1)} = \emptyset$ then we are done and we can tell apart any element of C via taking the dot product with v_1. Otherwise, notice that $0 \notin \mathrm{ch}(C^{(1)})$. Extending the separation theorem to arbitrary vector subspaces of $\mathbb{R}^n$ we can find an hyperplane $H^{(2)}$ of $H^{(1)}$ separating 0 and $\mathrm{ch}(C^{(1)})$, or equivalently, a vector $v_2 \in H^{(1)} \setminus \{0\}$ such that $v_2 \cdot x \geq 0$ for all $x \in \mathrm{ch}(C^{(1)})$ so that $H^{(2)} = v_1^{\perp} \cap v_2^{\perp}$. Again, if for some $\mu \in \mathbb{Z}^n \setminus \{0\}$ we have $v_1 \cdot \mu = 0$ and $v_2 \cdot \mu > 0$, then μ lies in $C^{(1)} \subseteq C$. We then define $C^{(2)} = H^{(2)} \cap C^{(1)}$ and ask whether $C^{(2)} = \emptyset$. If true, we can tell apart vectors in C via first taking their dot product with v_1 and, in case of a 0, subsequently taking the product with v_2, which would suffice. If not true, we continue our inductive process until we stop. It will have to stop necessarily when we find n vectors $v_1, v_2, \ldots, v_n$: they are pairwise orthogonal by construction, so by linear independence they form a basis of $\mathbb{R}^n$ and we would have $C^{(n)} = H^{(n)} \cap C^{(n-1)} = \left(v_1^\perp \cap v_2^\perp \cap \cdots \cap v_n^\perp\right) \cap C^{(n-1)} \subseteq \{0\} \cap C = \emptyset$. In any case, we could complete the orthogonal basis without affecting the weight order.