3. The grevlex order.

(a) Prove that grevlex is a monomial ordering

(b) Prove that the grevlex orderings on $\mathbf{F}[x_1, x_2]$ coincide wi the grlex orderings, bu the grevlex orderings on $\mathbf{F}[x_1, x_2, x_3]$ are not grlex orderings.

(c) Order the monomials in lex, grlex, and grevlex.

(a) Grevlex is a monomial ordering.

Proof. (Every two monomials are comparable:) Let $m_1 = x_1^{a_1} \ldots x_n^{a_n}$, $m_2 = x_1^{b_1} \ldots x_n^{b_n}$ and $x_1 > x_2 > \cdots > x_n$. First we'll show that every two monomials are comparable. Suppose $m_1 \neq m_2$. Then either deg $m_1 \neq$ deg m_2 or deg $m_1 =$ deg m_2 and there exists an i such that $a_i \neq b_i$ and $a_{i+1} = b_{i+1}, \ldots, a_n = b_n$. So either $m_1 > m_2$ or $m_2 > m_1$.

(Transitivity:)Suppose that $m_1 > m_2 > m_3$ with m_1, m_2 as before and $m_3 = x_1^{c_1} \ldots x_n^{c_n}$. If deg $m_1 >$ deg m_2 or deg $m_2 >$ deg m_3 then deg $m_1 >$ deg m_3 and $m_1 > m_3$. If not then deg $m_1 =$ deg $m_2 =$ deg m_3. Since $m_1 > m_2$ there exists an i such that $a_i < b_i$ and $a_{i+1} = b_{i+1}, \ldots, a_n = b_n$. Similarly there exits a j such that $b_j < c_j$ and $b_{j+1} = c_{j+1}, \ldots, b_n = c_n$. If $i \leq j$ then $a_j \leq b_j < c_j$ and $a_{j+1} = c_{j+1}, \ldots a_n = c_n$. If $i > j$ then $a_i < b_i = c_i$ and $a_{i+1} = c_{i+1}, \ldots a_n = c_n$. So $m_1 > m_3$.

$(1 < m)$: Every monomial has a degree greater than 0.

($m_1 > m_2 \to mm_1 > mm_2$): Let m_1, m_2 be as above and $m = x_1^{d_1} \dots x_n^{d_n}$. Assume that $m_1 > m_2$. If $\deg m_1 > \deg m_2$ then $\deg mm_1 = \deg m + \deg m_1 > \deg m + \deg m_2 = \deg mm_2$. If $\deg m_1 = \deg m_2$ then there exists an i such that $a_i < b_i$ and $a_{i+1} = b_{i+1}, \dots, a_n = b_n$. Then $mm_1 = x_1^{a_1+d_1} \dots x_n^{a_n+d_n}$, $mm_2 = x_1^{b_1+d_1} \dots x_n^{b_n+d_n}$ and $a_i + d_i < b_i + d_i$ and $a_{i+1}+d_{i+1} = b_{i+1}+d_{i+1}, \dots, a_n+d_n = b_n+d_n$. So $mm_1 > mm_2$. □

(b) Grevlex coincides with grlex on $\mathbf{F}[x_1, x_2]$.

Proof. Let $m_1 = x_1^{a_1} x_2^{a_2}$ and $m_2 = x_1^{b_1} x_2^{b_2}$. Assume that $m_1 \neq m_2$. If $\deg m_1 > \deg m_2$ then $m_1 > m_2$ in both grevlex and grlex. Suppose $\deg m_1 = \deg m_2$. Then either $a_1 > b_1$ or $a_1 < b_1$. Suppose that $a_1 > b_1$. Then $m_1 > m_2$ in grlex. But since $a_1 > b_1$ we must have $a_2 < b_2$ so $m_1 > m_2$ in grevlex. This doesn't work in $\mathbf{F}[x, y, z]$. Take for example $m_1 = x^3 y z^2$ and $m_2 = x y^4 z$. Then $m_1 > m_2$ in grlex but $m_2 > m_1$ in grevlex. □

(c) *Proof.* In lex: $x^3 y > x^3 z^2 > x^2 y^2 z > x^2 y z^2 > x^2 z^2 > x^2 z > x^2 > x y^2 z$. In grlex: $x^3 z^2 > x^2 y^2 z > x^2 y z^2 > x^3 y > x^2 z^2 > x y^2 z > x^2 z > x^2$. In grevlex: $x^2 y^2 z > x^3 z^2 > x^2 y z^2 > x^3 y > x y^2 z > x^2 z^2 > x^2 z > x^2$. □