combinatorial commutative algebra

federico ardila

san francisco state university
universidad de los andes

homework four due: wed. mar. 18 (sf) / vie. 20 de mar. (bog)

Note. You are encouraged to work together on the homework, but you must state who you worked with. You *must* write your solutions independently and in your own words. If you are turning in your homework by email, please send it to `cca.acc.cca@gmail.com`.

1. Two tricks for computing h-vectors. Prove Mark Skandera's and Richard Stanley's tricks for computing the h-vector of a simplicial complex, as explained in class.
2. Computing homology. Compute the homology groups for the simplicial complex on $\{a, b, c, d, e\}$ whose maximal faces are $\{a, b, c\}, \{a, c, d\}, \{a, d, e\}, \{b, e\}$.
3. On the dictionary between squarefree monomial ideals and simplicial complexes. (20 points)
   (a) What is the simplicial complex Δ corresponding to the square-free monomial ideal $I = \langle x_1y_1, x_2y_2, x_3y_3\rangle$ in $R = \mathbb{F}[x_1, x_2, x_3, y_1, y_2, y_3]$?
   (b) Compute the h-vector of Δ. Why should you expect it to be symmetric?
   (c) Compute the (coarse) Hilbert series of R/I.
   (d) Compute the minimal free resolution of I.
   (e) For each non-zero Betti number, draw the corresponding Koszul simplicial complex.
4. The h-vector of a join. If Δ_1 and Δ_2 are simplicial complexes on disjoint sets E_1 and E_2, we define the *join* $\Delta_1 * \Delta_2$ to be the simplicial complex on $E_1 \cup E_2$ whose faces are the sets $A_1 \cup A_2$ with $A_1 \in \Delta_1$ and $A_2 \in \Delta_2$. Compute the h-vector of $\Delta_1 * \Delta_2$ in terms of the h-vectors of Δ_1 and Δ_2.
5. (Bonus: 10 points) A more interesting Hilbert series. Compute the (coarse) Hilbert series of
$$\mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_n]/\langle x_1y_1, x_2y_2, \ldots, x_ny_n\rangle.$$
6. (Bonus: 5 points) 'Tor' and torsion. Suppose a is not a zero divisor in a commutative ring R, and M is an R-module. Prove that
$$\mathrm{Tor}_1(R/\langle a\rangle, M) \cong \{m \in M \mid am = 0\}.$$
7. Regarding that final project... In a *short* paragraph, tell me your thoughts about the final project. What topics are you considering? Are you interested in surveying an existing topic, or will you try to obtain new results? What are three specific references (articles or sections of books) that you might use?