combinatorial commutative algebra

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homework two due: wed. feb. 18 (sf) / vie. 20 de feb. (bog)

Note. You are encouraged to work together on the homework, but you must state who you worked with. You *must* write your solutions independently and in your own words. If you are turning in your homework by email, please send it to `cca.acc.cca@gmail.com`.

1. A proof of Hilbert's basis theorem using Gröbner bases.
   (a) Let P be the partially ordered set of (monic) monomials in the variables $x_1, \ldots, x_n$, where $m_1 \leq m_2$ if and only if m_1 divides m_2. Give a combinatorial proof (without using Hilbert's basis theorem) of the fact that any subset of P has a finite number of minimal elements.
   (b) Use part (a) to show that every ideal of $\mathbb{F}[x_1, \ldots, x_n]$ has a finite Gröbner basis, and hence is finitely generated.
2. Minimal Gröbner bases. Fix an ideal I of $\mathbb{F}[x_1, \ldots, x_n]$ and a monomial order.
   (a) Prove that $\{g_1, \ldots, g_m\} \subset I$ is a minimal Gröbner basis for I if and only if the multiset $\{\text{in}(g_1), \ldots, \text{in}(g_m)\}$ is a minimal generating set for $\text{in}(I)$.
   (b) Prove that any two minimal Gröbner bases for I have the same size and the same set of leading terms.
3. Gröbner basis computations. Perform the following computations using Gröbner bases with respect to the lexicographic order $x > y$ (or $x > y > z$) in $\mathbb{C}[x, y]$ (or $\mathbb{C}[x, y, z]$):
   (a) Determine whether $x^6 - x^5y$ is in $\langle x^3 - y, x^2y - y^2, xy^2 - y^2, y^3 - y^2\rangle$.
   (b) Determine whether $\langle x^3 - yz, yz + y\rangle = \langle x^3z + x^3, x^3 + y\rangle$.
   (c) Solve the system of equations $x^2 - yz = 3$, $y^2 - xz = 4$, $z^2 - xy = 5$.
   (d) Compute $\langle x^3y - xy^2 + 1, x^2y^2 - y^3 - 1\rangle \bigcap \langle x^2 - y^2, x^3 + y^3\rangle$.
   (e) Compute the syzygies between the polynomials x^2, y^2, and $xy + yz$.
4. Hilbert series. Let $I = \langle x_1x_3, x_1x_4, x_2x_4\rangle$ in $\mathbb{F}[x_1, x_2, x_3, x_4]$, and let I_d be the $\mathbb{F}$-vector space of homogeneous polynomials of degree d in I. Compute the *Hilbert function* $H_I : \mathbb{N} \to \mathbb{N}$ and *Hilbert series* $H(I; x) \in \mathbb{C}[[x]]$, which are defined to be:
   $$H_I(d) := \dim_{\mathbb{F}} I_d, \qquad \text{and} \qquad H(I; x) := \sum_{n \geq 0} H(d)x^d,$$
   respectively.
5. Computing free resolutions. Let $R = \mathbb{F}[x_1, \ldots, x_n]$ and $I = \langle x_1, \ldots, x_n\rangle$. Compute a finite free resolution for the R-module R/I.
6. (Bonus: 10 points) Free resolutions over other rings. Let $m \leq n$ be positive integers, and let $R = \mathbb{F}[x]/\langle x^n\rangle$.
   (a) Compute a free resolution for the R-module $R/\langle x^m\rangle$.
   (b) Prove that the only R-modules which have a finite free resolution are the free R-modules.