combinatorial commutative algebra

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homework one (due wed. feb. 4 (sf) or fri. feb. 6 (bog))

Note. You are encouraged to work together on the homework, but you must state who you worked with. You *must* write your solutions independently and in your own words. You may turn in your homework a paper copy in class in SF, or by email before midnight at `cca.acc.cca@gmail.com`.

0. Register in the online discussion forum, following the instructions on the course website.

1. A non-Noetherian ring. Let R be the ring of continuous real valued functions on $[0,1]$. Find an ideal of R which is not finitely generated.

2. Finite intervals. Say a monomial ordering of $\mathbb{F}[x_1,\ldots,x_n]$ has the *finite interval property* if for any monomials a,b with $a<b$ there exist finitely many monomials c such that $a<c<b$. Does the grlex order have the finite interval property? Does every monomial ordering have the finite interval property?

3. The grevlex order. The *grevlex* (graded reverse lexicographic) order of the monomials in $\mathbb{F}[x_1,\ldots,x_n]$ is defined by first choosing an ordering of the variables, say $x_1>\cdots>x_n$ and then defining $m_1\geq m_2$ for monomials m_1,m_2 if either

   o $\deg m_1>\deg m_2$, or

   o $\deg m_1=\deg m_2$ and the first exponent of $x_n,x_{n-1},\ldots,x_1$ (in that order) where m_1 and m_2 differ is smaller in m_1.

   (a) Prove that grevlex is a monomial ordering.

   (b) Prove that the grevlex orderings on $\mathbb{F}[x_1,x_2]$ coincide with the grlex orderings, but the grevlex orderings on $\mathbb{F}[x_1,x_2,x_3]$ are not grlex orderings.

   (c) Using $x>y>z$, order the monomials $x^2z, x^2y^2z, xy^2z, x^3y, x^3z^2, x^2, x^2yz^2, x^2z^2$ with respect to lex, with respect to grlex, and with respect to grevlex.

4. A geometric characterization of term orders. Let $v_1,\ldots,v_n$ form a basis of $\mathbb{R}^n$. Define the *weight order* $\geq$ on the monomials of $\mathbb{F}[x_1,\ldots,x_n]$ by setting $m_1>m_2$ if and only if for some $t\leq n$ we have

$$\begin{aligned} v_i\cdot \text{multideg}(m_1) &= v_i\cdot \text{multideg}(m_2) \text{ for } 1\leq i\leq t-1, \text{ and}\\ v_t\cdot \text{multideg}(m_1) &> v_t\cdot \text{multideg}(m_2), \end{aligned}$$

   where $\cdot$ denotes the dot product.

   (a) If $v_i=(0,\ldots,0,1,0,\ldots,0)$ (with a 1 in the ith coordinate) for $1\leq i\leq n$, show that the weight order is the lexicographic order with $x_1>\cdots>x_n$.

   (b) If $v_1=(1,\ldots,1)$, and $v_i=(1,\ldots,1,-n+i-1,0,\ldots 0)$ (with $-n+i-1$ in the $(n-i+2)$nd coordinate) for $2\leq i\leq n$, show that the resulting order is the grevlex order with $x_1>\cdots>x_n$.

   (c) Describe the n-tuples of vectors $v_1,\ldots,v_n$ such that the weight order is a monomial ordering. (Hint: see forum.)

   (d) (10 bonus points) Show that any monomial ordering on $\mathbb{F}[x_1,\ldots,x_n]$ may be obtained as a weight order, using an appropriate choice of vectors.

5. Polynomials in initial ideals. Fix a monomial ordering $<$ on $R=\mathbb{F}[x_1,\ldots,x_n]$, and a Gröbner basis $\{g_1,\ldots,g_m\}$ for an ideal I of R. Prove that a polynomial $h\in R$ is in $\text{in}_<(I)$ if and only if it is a sum of monomial terms, each one of which is divisible by $\text{in}_<(g_i)$ for some i.