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homework four

. due thursday oct 23 at the beginning of class.

Note. You are encouraged to work together on the homework, but you must state who you worked with **in each problem**. You must write your solutions independently and in your own words. (I recommend putting away the notes from your discussions with others, and reproducing the solutions by yourself.)

1. (On formal power series.) Let $\mathbb{C}[[x]]$ be the ring of formal power series with complex coefficients.
   (a) Prove that $\mathbb{C}[[x]]$ is an integral domain.
   (b) Prove that $a_0 + a_1x + a_2x^2 + \cdots$ is invertible in $\mathbb{C}[[x]]$ if and only if $a_0 \neq 0$.
2. (Personalized Catalan problem.) Prove that the combinatorial objects assigned to you are enumerated by the Catalan numbers.
3. (A quadratic recurrence.) Find the unique sequence $a_0, a_1, a_2, \ldots$ satisfying
$$\sum_{k=0}^{n} a_k a_{n-k} = 1.$$
   for any $n \geq 0$.
4. (A functional equation.) Find (and prove the uniqueness of) the formal power series $B(x)$ such that
$$[x^n](B(x))^{n+1} = 1$$
   for all $n \geq 0$. (Hint: Use Lagrange inversion.)
5. (Ordered set partitions.) Let l_n be the number of ways of partitioning the set $[n]$ into non-empty blocks, putting the blocks in a linear order, and putting the elements of each block in a linear order.
   (a) Use generating functions to compute l_n.
   (b) Give a combinatorial proof.
6. (Bonus problem: Paths in a $2 \times n$ grid.) Consider a grid of height 2 and length n. Find the number of paths of length n which start in the lower left corner of the grid, which consist of unit steps up, down, or right, and which never retrace their steps.