algebraic combinatorics

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homework three . due tuesday oct 7 at 11:59 p.m.

Note. You are encouraged to work together on the homework, but you must state who you worked with **in each problem**. You must write your solutions independently and in your own words. (I recommend putting away the notes from your discussions with others, and reproducing the solutions by yourself.)

1. (Paths in a $1 \times n$ grid.) Consider a grid of height 1 and length n. Find the number of paths of length n which start in the lower left corner of the grid, which consist of unit steps up, down, or right, and which never retrace their steps.
2. (Partitions with restrictions.) Let j and k be fixed positive integers.
   (a) Let $p_{\leq j, \leq k}(n)$ be the number of partitions of n into at most j parts, all of which are at most k. Prove that
   $$\sum_{n \geq 0} p_{\leq j, \leq k}(n) x^n = \frac{(1-x)(1-x^2)\cdots(1-x^{j+k})}{(1-x)(1-x^2)\cdots(1-x^j) \cdot (1-x)(1-x^2)\cdots(1-x^k)}$$
   Conclude that the right hand side is actually a polynomial in x.
   (b) This formula implies that $p_{\leq j, \leq k}(n) = p_{\leq k, \leq j}(n)$. Give a combinatorial proof.
   (c) (Bonus.) Prove that if $x = q$ is a prime power and $\mathbb{F}_q$ is the finite field of q elements, then the above expression equals the number of j-dimensional subspaces of $\mathbb{F}_q^{j+k}$. Explain algebraically why this answer is symmetric in j and k.
3. (Restricted compositions) Find the number of compositions of n having an even number of even parts (and any number of odd parts).
4. (Occurrences of a part in compositions) Let $n \geq k$ be fixed positive integers. Find the total number of times that the number k appears as a summand among all the compositions of n.
5. (Two statistics on Dyck paths) For a Dyck path P, let $a(P)$ be the number of upsteps before the first downstep of P, and let $b(P)$ be the number of times that P returns to the x-axis after leaving it for the first time. Let
   $$T_n(x, y) = \sum_P x^{a(P)} y^{b(P)}.$$
   summing over all Dyck paths of length n.
   (a) Compute the generating function $T(x, y, z) = \sum_{n \geq 0} T_n(x, y) z^n$ in terms of the generating function $C(z) = \sum_{n \geq 0} C_n z^n$ for Catalan numbers.
   (b) Conclude that $T_n(x, y) = T_n(y, x)$. ((Bonus.) Give a combinatorial proof.)
6. (Bonus problem: A "Hadamard" product of two sequences.) Let j and k be positive integers. Define sequences a_n and b_n by
   $$\sum_{n \geq 0} a_n x^n = \frac{1}{1 - jx - x^2}, \qquad \sum_{n \geq 0} b_n x^n = \frac{1}{1 - kx - x^2},$$
   Compute
   $$\sum_{n \geq 0} a_n b_n x^n.$$
   (Hint: think combinatorially.)