TWO COUNTEREXAMPLES FOR POWER IDEALS OF HYPERPLANE ARRANGEMENTS.

FEDERICO ARDILA AND ALEXANDER POSTNIKOV

Abstract. We disprove Holtz and Ron's conjecture that the power ideal $C_{\mathcal{A},-2}$ of a hyperplane arrangement $\mathcal{A}$ (also called the internal zonotopal space) is generated by $\mathcal{A}$-monomials. We also show that, in contrast with the case $k \geq -2$, the Hilbert series of $C_{\mathcal{A},k}$ is not determined by the matroid of $\mathcal{A}$ for $k \leq -6$.

Remark. This note is a corrigendum to our article [1], and we follow the notation of that paper.

1. Introduction.

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a hyperplane arrangement in a vector space V; say $H_i = \{x \mid l_i(x) = 0\}$ for some linear functions $l_i \in V^*$. Call a product of (possibly repeated) l_is an $\mathcal{A}$-*monomial* in the symmetric algebra $\mathbb{C}[V^*]$. Let Lines($\mathcal{A}$) be the set of lines of intersection of the hyperplanes in $\mathcal{A}$. For each $h \in V$ with $h \neq 0$, let $\rho_{\mathcal{A}}(h)$ be the number of hyperplanes in $\mathcal{A}$ not containing h. Let $\rho = \rho(\mathcal{A}) = \min_{h \in V}(\rho_{\mathcal{A}}(h))$. For all integers $k \geq -(\rho+1)$, consider the *power ideals*:

$$I_{\mathcal{A},k} := \left\langle h^{\rho_{\mathcal{A}}(h)+k+1} \mid h \in V, h \neq 0 \right\rangle, \quad I'_{\mathcal{A},k} := \left\langle h^{\rho_{\mathcal{A}}(h)+k+1} \mid h \in \mathrm{Lines}(\mathcal{A}) \right\rangle$$

in the symmetric algebra $\mathbb{C}[V]$. It is convenient to regard the polynomials in $I_{\mathcal{A},k}$ as differential operators, and to consider the space of solutions to the resulting system of differential equations:

$$C_{\mathcal{A},k} = I_{\mathcal{A},k}^{\perp} := \left\{ f(x) \in \mathbb{C}[V^*] \mid h\left(\frac{\partial}{\partial x}\right)^{\rho_{\mathcal{A}}(h)+k+1} f(x) = 0 \text{ for all } h \neq 0 \right\}$$

which is known as the *inverse system* of $I_{\mathcal{A},k}$. Define $C'_{\mathcal{A},k}$ similarly. These objects arise naturally in numerical analysis, algebra, geometry, and combinatorics. For references, see [1, 3].

One important question is to compute the Hilbert series of these spaces of polynomials, graded by degree, as a function of combinatorial invariants of $\mathcal{A}$. Frequently, the answer is expressed in terms of the Tutte polynomial of $\mathcal{A}$. This has been done successfully in many cases. One strategy used independently by different authors has been to prove the following:

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(i) There is a spanning set of $\mathcal{A}$-monomials for $C_{\mathcal{A},k}$.
(ii) There is an exact sequence $0 \to C_{\mathcal{A}\backslash H,k}(-1) \to C_{\mathcal{A},k} \to C_{\mathcal{A}/H,k} \to 0$ of graded vector spaces.
(iii) Therefore, the Hilbert series of $C_{\mathcal{A},k}$ is an evaluation of the Tutte polynomial of $\mathcal{A}$.

Here $\mathcal{A}\backslash H$ and $\mathcal{A}/H$ are the deletion and contraction of H, respectively.

For $k \geq -1$, this method works very nicely. Dahmen and Michelli [2] were the first ones to do this for $C'_{\mathcal{A},-1}$. Postnikov-Shapiro-Shapiro [5] did it for $C_{\mathcal{A},0}$, while Holtz and Ron [3] did it for $C'_{\mathcal{A},0}$. In [1] we did it for $C_{\mathcal{A},k}$ for all $k \geq -1$, and showed that $C'_{\mathcal{A},0} = C_{\mathcal{A},0}$ and $C'_{\mathcal{A},-1} = C_{\mathcal{A},-1}$.

For $k \leq -3$ this approach does not work in full generality. In [1] we showed that (i) is false in general for $C_{\mathcal{A},k}$, and left (ii) and (iii) open, suggesting the problem of measuring $C_{\mathcal{A},k}$. For $k \leq -6$, (ii) and (iii) are false, as we will show in Propositions 4 and 5, respectively. In fact, we will see that the Hilbert series of $C_{\mathcal{A},k}$ is not even determined by the matroid of $\mathcal{A}$.

The intermediate cases are interesting and subtle, and deserve further study; notably the case $k = -2$, which Holtz and Ron call the *internal zonotopal space*. In [3] they proved (ii) and (iii) and conjectured (i) for $C'_{\mathcal{A},-2}$. In [1, Proposition 4.5.3] – a restatement of Holtz and Ron's Conjecture 6.1 in [3] – we put forward an incorrect proof of this conjecture; the last sentence of our argument is false. In fact their conjecture is false, as we will see in Proposition 2.

2. The case $k = -2$: internal zonotopal spaces.

Before showing why Holtz and Ron's conjecture is false, let us point out that the remaining statements about $C_{\mathcal{A},-2}$ that we made in [1] are true. The easiest way to derive them is to prove that $C_{\mathcal{A},-2} = C'_{\mathcal{A},-2}$, and simply note that Holtz and Ron already proved those statements for $C'_{\mathcal{A},-2}$:

Lemma 1. *We have $C_{\mathcal{A},k} = C'_{\mathcal{A},k}$ for any k with $-(\rho+1) \leq k \leq 0$.*

Proof. By [1, Theorem 4.17] we have $I_{\mathcal{A},0} = I'_{\mathcal{A},0}$, so it suffices to show that $I_{\mathcal{A},j} = I'_{\mathcal{A},j}$ implies that $I_{\mathcal{A},j-1} = I'_{\mathcal{A},j-1}$ as long as these ideals are defined. If $I_{\mathcal{A},j} = I'_{\mathcal{A},j}$, then for any $h \in V\backslash\{0\}$ we have $h^{\rho_{\mathcal{A}}(h)+j+1} = \sum f_i h_i^{\rho_{\mathcal{A}}(h_i)+j+1}$ for some polynomials f_i, where the h_is are the lines of the arrangement. As long as the exponents are positive, taking partial derivatives in the direction of h gives $h^{\rho_{\mathcal{A}}(h)+j} = \sum g_i h_i^{\rho_{\mathcal{A}}(h_i)+j}$ for some polynomials g_i. □

The following result shows that (i) does not hold for $C_{\mathcal{A},-2}$.

Proposition 2. [3, Conjecture 6.1] is false*: The "internal zonotopal space" $C_{\mathcal{A},-2}$ is not necessarily spanned by $\mathcal{A}$-monomials.*

Proof. Let $\mathcal{H}$ be the hyperplane arrangement in $\mathbb{C}^4$ determined by the linear forms $y_1, y_2, y_3, y_1 - y_4, y_2 - y_4, y_3 - y_4$. We have

$$I'_{\mathcal{H},-2} = \langle x_1^1, x_2^1, x_3^1, (\epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_4 + x_4)^2\rangle = \langle x_1, x_2, x_3, x_4^2\rangle$$

as $\epsilon_1, \epsilon_2, \epsilon_3$ range over $\{0,1\}$. The other generators of $I_{\mathcal{H},-2}$ are of degree at least 3, and are therefore in $I'_{\mathcal{H},-2}$ already, so

$$I_{\mathcal{H},-2} = \langle x_1, x_2, x_3, x_4^2\rangle, \qquad C_{\mathcal{H},-2} = \mathrm{span}(1, y_4).$$

Therefore $C_{\mathcal{H},-2}$ is not spanned by $\mathcal{H}$-monomials. $\square$

As Holtz and Ron pointed out, if [3, Conjecture 6.1] had been true, it would have implied [3, Conjecture 1.8], an interesting spline-theoretic interpretation of $C_{\mathcal{A},-2}$ when $\mathcal{A}$ is unimodular. The arrangement above is unimodular, but it does not provide a counterexample to [3, Conjecture 1.8]. In fact, Matthias Lenz [4] has recently put forward a proof of this weaker conjecture.

3. The case $k \leq -6$

In this section we show that when $k \leq -6$, the Hilbert series of $C_{\mathcal{A},k}$ is not a function of the Tutte polynomial of $\mathcal{A}$. In fact, it is not even determined by the matroid of $\mathcal{A}$. Recall that $\rho = \rho(\mathcal{A}) := \min_{h\in V}(\rho_{\mathcal{A}}(h))$. Say $h \in V$ is *large* if it is on the maximum number of hyperplanes, so $\rho_{\mathcal{A}}(h) = \rho$.

Lemma 3. *The degree* 1 *component of* $C_{\mathcal{A},-\rho}$ *is*

$$(C_{\mathcal{A},-\rho})_1 = (\mathrm{span}\{h \in V \,:\, h \textit{ is large}\})^{\perp}$$

in V^*.

Proof. An element f of $C_{\mathcal{A},-\rho}$ needs to satisfy the differential equations $h\,(\partial/\partial x)^{\rho_{\mathcal{A}}(h)-\rho+1}\, f(x) = 0$ for all non-zero vectors $h \in V$. If f is linear, then this condition is trivial unless h is large; and in that case it says that $f \perp h$. $\square$

Proposition 4. *For* $k \leq -6$*, the Hilbert series of* $C_{\mathcal{A},k}$ *is not determined by the matroid of* $\mathcal{A}$*.*

Proof. First assume $k = -2m$. Let L_1, L_2, L_3 be three different lines through 0 in $\mathbb{C}^3$ and consider an arrangement $\mathcal{A}$ of $3m$ (hyper)planes consisting of m generically chosen planes $H_{i1}, \ldots, H_{im}$ passing through L_i for $i = 1,2,3$. Then $\rho = 2m$ and the only large lines are L_1, L_2, and L_3. Therefore $\dim(C_{\mathcal{A},-2m})_1$ equals 1 if L_1, L_2, L_3 are coplanar, and 0 otherwise. However, the matroid of $\mathcal{A}$ does not know whether L_1, L_2, L_3 are coplanar.

More precisely, consider two versions $\mathcal{A}_1$ and $\mathcal{A}_2$ of the above construction; in $\mathcal{A}_1$ the lines L_1, L_2, L_3 are coplanar, and in $\mathcal{A}_2$ they are not. Notice that $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same matroid: the rank 3 matroid whose non-bases are the triples $\{H_{ia}, H_{ib}, H_{ic}\}$ for $1 \leq i \leq 3$ and $1 \leq a < b < c \leq m$. However, $\dim(C_{\mathcal{A}_1,-2m})_1 \neq \dim(C_{\mathcal{A}_2,-2m})_1$.

The case $k = -2m-1$ is similar. It suffices to add a generic plane to the previous arrangements. $\square$

Proposition 5. *For $k \leq -6$, the sequence of graded vector spaces*

$$0 \to C_{\mathcal{A}\backslash H,k}(-1) \to C_{\mathcal{A},k} \to C_{\mathcal{A}/H,k} \to 0$$

of [1, Proposition 4.4.1] *is not necessarily exact, even if H is neither a loop nor a coloop.*

Proof. We will not need to recall the maps that define this sequence; we will simply show an example where right exactness is impossible because $\dim(C_{\mathcal{A},k})_1 = 0$ and $\dim(C_{\mathcal{A}/H,k})_1 = 1$. We do this in the case $k = -2m$; the other one is similar.

Consider the arrangement $\mathcal{A} = \mathcal{A}_2$ of the proof of Proposition 4 and the plane $H = H_{11}$. We have $\dim(C_{\mathcal{A},-2m})_1 = 0$. In the contraction $\mathcal{A}/H$, the planes $H_{12}, \ldots, H_{1m}$ become the same line L_1 in H, while the other $2m$ planes of $\mathcal{A}$ become generic lines in H. Therefore $\rho(\mathcal{A}\backslash H) = 2m$ and $(C_{\mathcal{A}/H,-2m})_1 = L_1^{\perp}$ in H^*, which is one-dimensional. □

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Department of Mathematics, San Francisco State University, 1600 Holloway Ave, San Francisco, CA 94110, USA.
E-mail address: `federico@sfsu.edu`

Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Ave, Cambridge, MA 02139, USA.
E-mail address: `apost@math.mit.edu`