The harmonic polytope

Federico Ardila* and Laura Escobar†

Abstract

We study the harmonic polytope, which arose in Ardila, Denham, and Huh's work on the Lagrangian geometry of matroids. We show that it is a $(2n-2)$-dimensional polytope with $(n!)^2(1+\frac{1}{2}+\cdots+\frac{1}{n})$ vertices and 3^n-3 facets. We give a formula for its volume: it is a weighted sum of the degrees of the projective varieties of all the toric ideals of connected bipartite graphs with n edges; or equivalently, a weighted sum of the lattice point counts of all the corresponding trimmed generalized permutahedra.

1 Introduction

Motivated by the Lagrangian geometry of conormal varieties in classical algebraic geometry, Ardila, Denham, and Huh [1] introduced the *conormal fan* $\Sigma_{\mathsf{M},\mathsf{M}^\perp}$ of a matroid M – a Lagrangian analog of the better known Bergman fan Σ_{M} [2]. They used the conormal fan $\Sigma_{\mathsf{M},\mathsf{M}^\perp}$ to give new geometric interpretations of the Chern-Schartz-MacPherson cycle of a matroid – introduced by López de Medrano, Rincón, and Shaw in [8] – and of the h-vectors of the broken circuit complex $BC(\mathsf{M})$ and independence complex $I(\mathsf{M})$ of M. This geometric framework allowed them to prove that these vectors are log-concave, as conjectured by Brylawski and Dawson [3, 6] in the 1980s.

In their construction of the conormal fan $\Sigma_{\mathsf{M},\mathsf{M}^\perp}$, Ardila, Denham, and Huh encountered two polytopes associated to a positive integer n: the *harmonic polytope* $H_{n,n}$ and the *bipermutohedron* $\Pi_{n,n}$; the first is a Minkowski summand of the second. This paper studies the harmonic polytope $H_{n,n}$; its name derives from the fact that its number of vertices is $(n!)^2 H_n$ where $H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$ is the nth harmonic sum.

The harmonic polytope has nice vertex and inequality descriptions, shown in Propositions 2.5 and 2.7. Our main result is Theorem 1.1, which shows that the volume of $H_{n,n}$ is a weighted sum of the degrees of the toric ideals of all bipartite multigraphs on n edges; or equivalently, a weighted sum of the lattice point count of all the corresponding trimmed generalized permutahedra.

Theorem 1.1. *The volume of the harmonic polytope is*

$$\mathsf{Vol}(H_{n,n}) = \sum_{\Gamma} \frac{i(P_\Gamma^-)}{(v(\Gamma)-2)!} \prod_{v\in V(\Gamma)} \mathsf{deg}(v)^{\mathsf{deg}(v)-2} = \sum_{\Gamma} \frac{\mathsf{deg}(X_\Gamma)}{(v(\Gamma)-2)!} \prod_{v\in V(\Gamma)} \mathsf{deg}(v)^{\mathsf{deg}(v)-2},$$

summing over all connected bipartite multigraphs Γ on edge set $[n]$. Here $i(P_\Gamma^-)$ is the number of lattice points in the trimmed generalized permutahedron P_Γ^- of Γ, X_Γ is the projective embedding of the toric variety of Γ given by the toric ideal of Γ, $V(\Gamma)$ is the set of vertices of Γ, and $v(\Gamma) := |V(\Gamma)|$.

*San Francisco State University, Universdidad de Los Andes; federico@sfsu.edu. Partially supported by NSF grant DMS-1855610 and Simons Fellowship 613384.

†Washington University in St. Louis; laurae@wustl.edu. Partially supported by NSF grant DMS-1855598.

In the statement above and throughout the paper, volumes are normalized so that any primitive lattice parallelotope has volume 1. For more details, see the introduction to Section 3.

The harmonic polytope can be expressed as a Minkowski sum of simplices, and its volume is a sum of the associated mixed volumes. We show that the non-zero mixed volumes are in bijection with the pairs of forests on $[n]$ whose union is connected. We count them in Proposition 3.11, by computing in the Möbius algebra of the partition lattice.

2 The harmonic polytope

Let n be a positive integer and let $[n] := \{1, \ldots, n\}$. Consider two copies of $\mathbb{R}^n$ with standard bases $\{\mathsf{e}_i : i \in [n]\}$ and $\{\mathsf{f}_i : i \in [n]\}$, respectively. For any subset S of $[n]$, we write

$$\mathbf{e}_S = \sum_{i\in S} \mathbf{e}_i, \qquad \mathbf{f}_S = \sum_{i\in S} \mathbf{f}_i.$$

We also consider the $(n-1)$-dimensional vector space $\mathsf{N}_n := \mathbb{R}^n/\mathbb{R}\mathsf{e}_{[n]}$.

The (inner) normal fan $\mathcal{N}(P)$ of a polytope $P \subset \mathbb{R}^d$ is a complete fan in the dual space $(\mathbb{R}^d)^*$ whose cones are

$$\mathcal{N}(P)_Q := \{w \in (\mathbb{R}^d)^* : P_w \supseteq Q\}$$

for each nonempty face Q of P, where $P_w = \{x \in P : w(x) = \min_{y\in P} w(y)\}$ is the w-minimal face of P. The face poset of the normal fan of P is isomorphic to the reverse of the face poset of P.

The normal fan of the permutohedron

$$\Pi_n = \mathsf{conv}\Big\{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \text{ is a permutation of } [n]\Big\} \subseteq \mathbb{R}^n.$$

is the *permutohedral fan* $\Sigma_n \subset \mathsf{N}_n$, also known as the *braid fan* or the *type A Coxeter complex*. It is the complete simplicial fan in $\mathbb{R}^n$ whose chambers are separated by the n-dimensional *braid arrangement*, the real hyperplane arrangement in $\mathbb{R}^n$ consisting of the $\binom{n}{2}$ hyperplanes

$$z_i = z_j, \quad \text{for distinct elements } i \text{ and } j \text{ of } [n].$$

The face of the permutohedral fan containing a given point z in its relative interior is determined by the relative order of its homogeneous coordinates $(z_1, \ldots, z_n)$.

Let D_n be the $(n-1)$-dimensional simplex,

$$D_n := \mathsf{conv}\Big\{\mathsf{e}_i + \mathsf{f}_i : i \in [n]\Big\} \subseteq \mathbb{R}^n \times \mathbb{R}^n.$$

The normal fan of the simplex D_n is the simplicial fan $\Delta_{n,n}$ whose n chambers are the cones

$$\mathscr{C}_k = \Big\{(z, w) \in \mathbb{R}^n \times \mathbb{R}^n \mid \min_{i\in[n]}(z_i + w_i) = z_k + w_k\Big\}.$$

Recall that the *Minkowski sum* and *Minkowski difference* of polytopes P and Q in $\mathbb{R}^d$ are

$$P + Q = \{p + q : p \in P, q \in Q\}, \qquad P - Q = \{r \in \mathbb{R}^d : r + Q \subseteq P\}.$$

The following polytope is our main object of study.

Definition 2.1. The *harmonic polytope* is the Minkowski sum

$$H_{n,n} := D_n + (\Pi_n \times \Pi_n) \subset \mathbb{R}^n \times \mathbb{R}^n.$$

The *harmonic fan* is its reduced normal fan $\mathcal{N}(H_{n,n})$ in $\mathsf{N}_n \times \mathsf{N}_n$.

Figure 1 shows the harmonic polytope $H_{2,2}$ and its reduced normal fan. The normal fan of $H_{n,n}$ is the coarsest common refinement of the normal fans of D_n and $\Pi_n \times \Pi_n$. Its lineality space is $\mathbb{R}\{\mathsf{e}_{[n]}, \mathsf{f}_{[n]}\}$.

2.1 The face structure of the harmonic polytope.

The face of the harmonic fan containing a point $(z, w) \in \mathsf{N}_n \times \mathsf{N}_n$ is determined by:
- the set of indices i for which the minimum of $z_i + w_i$ is attained,
- the **reverse**[1] relative order of the z_is, and
- the reverse relative order of the w_is.

Our next task is to characterize the triples that arise in this way.

Recall that an *ordered set partition* of $[n]$ is a sequence $\pi = E_1|\cdots|E_\ell$ such that $E_1 \cup \cdots \cup E_\ell = [n]$ and $E_i \cap E_j = \emptyset$ for all $i \neq j$. The *length* of π is $\ell(\pi) := \ell$. The ordered set partitions of $[n]$ form a poset under *adjacent refinement*, where $\pi \leq \pi'$ if every block of π' is a union of a set of consecutive blocks of π. For example $14|3|26|8|57 \leq 134|26|578$.

Definition 2.2. The poset of harmonic triples HT_n is defined as follows:

1. A *harmonic triple* $\tau = (K; \pi_1, \pi_2)$ on $[n]$ consists of a nonempty subset $K \subseteq [n]$ and two ordered set partitions π_1 and π_2 of $[n]$ such that
   (a) The restrictions $\pi_1|K$ and $\pi_2|K$ of π_1 and π_2 to K are opposite to each other, and
   (b) If $j \notin K$ appears in the same or a later block than $k \in K$ in one of the set partitions π_1 and π_2, then j must appear in an earlier block than k in the other set partition.
2. The *poset of harmonic triples* HT_n is defined by setting $(K; \pi_1, \pi_2) \leq (K'; \pi'_1, \pi'_2)$ if and only if $K \subseteq K'$, π_1 is an adjacent refinement of π'_1, and π_2 is an adjacent refinement of π'_2.
3. A *fine harmonic triple* is a minimal element of the poset HT_n. A *coarse harmonic triple* is a maximal element of $HT_n - \{\widehat{1}\}$.

Notice that the maximum element $\widehat{1}$ of HT_n is the triple $([n], [n], [n])$. The fine harmonic triples are the minimal elements, for which K consists of a single element k, and π_1 and π_2 only have blocks of size 1 – and hence may be thought of as permutations in one-line notation.

Example 2.3. Consider the triple $(\mathbf{3467}, \mathbf{4}5|8|2|1\mathbf{37}9|\mathbf{6}, \mathbf{6}|1|59|2\mathbf{37}|8|\mathbf{4})$, were we omit the brackets and write the elements of K in bold for easier readability. The reader is invited to verify that this triple satisfies the required conditions to be harmonic. On the other hand, $j = 1$ and $k = 3$ do not satisfy condition (b) in the non-harmonic triple $(\mathbf{3467}, \mathbf{4}5|8|2|1\mathbf{37}9|\mathbf{6}, \mathbf{6}|5|2\mathbf{37}|89|1\mathbf{4})$.

[1]Of course, this is the same information as the relative order of the z_is. We use the reverse order because it is consistent with our choice of working with inner normal fans.

Proposition 2.4. *The combinatorial structure of the harmonic fan $\mathcal{N}(H_{n,n})$ is as follows.*

1. *The faces of the harmonic fan are in bijection with the harmonic triples on $[n]$.*
2. *The dimension of the face labeled by $\tau = (K; \pi_1, \pi_2)$ is $\ell(\pi_1) + \ell(\pi_2) - \ell(\pi_1|K) - 1$.*
3. *Two faces F and F' of the harmonic fan satisfy $F \supseteq F'$ if and only if their harmonic triples satisfy $\tau \leq \tau'$ in HT_n.*

Proof. 1. Given a face F of the harmonic fan, we define the triple $\tau(F)$ as follows. Let (z, w) be an interior point of F. We let K be the set of indices k for which the minimum of $z_k + w_k$ is attained, π_1 be the partition encoding the reverse relative order of the z_is, and π_2 be the reverse relative order of the w_is. For example, we have, for the following face F,

$$\begin{array}{rl} z_k + w_k \text{ is minimum for } k = 3,4,6,7 & \\ z_6 < z_1 = z_3 = z_7 = z_9 < z_2 < z_8 < z_4 = z_5 & \mapsto \quad \tau(F) = (\mathbf{3467}, \mathbf{4}5|8|2|1\mathbf{37}9|\mathbf{6}, \mathbf{6}|1|59|2\mathbf{37}|8|\mathbf{4}). \\ w_4 < w_8 < w_2 = w_3 = w_7 < w_5 = w_9 < w_1 < w_6 & \end{array}$$

Since $z_k + w_k$ is constant for k in K, the relative order of the z_ks is exactly the opposite of the relative order of the w_ks, so (a) holds. Also, if $j \notin K$ appears in the same or a later block than $k \in K$ in, say the first set partition, then we have $z_k \geq z_j$. But then $z_k + w_k < z_j + w_j$ implies that $w_k < w_j$, so j must appear before k in the second set partition. Therefore (b) also holds.

Conversely, suppose $\tau = (K; \pi_1, \pi_2)$ is a harmonic triple, and let us construct a point (z, w) whose associated triple is τ. We begin by defining the values of z_k and w_k for $k \in K$. We let $z_k = a$ where k is in the ath block of $\pi_2|K$ and $w_k = b$ where k is in the bth block of $\pi_1|K$. Then the z_ks and w_ks are in the order specified by $\pi_1|K$ and $\pi_2|K$, respectively, and, since $\pi_1|K$ and $\pi_2|K$ are opposites of each other, $z_k + w_k = c$ where $\pi_1|K$ and $\pi_2|K$ have $c - 1$ blocks.

Now define the values of z_j for $j \notin K$ as follows. If j is in the same block of π_1 as $k \in K$ set $z_j = z_k$. Define the remaining entries z_j to have the order stipulated by π_1, while making each one of them very large – say, within a small $\epsilon > 0$ of the first entry z_k such that $z_k > z_j$, if there is one. For example, for the triple $\tau = (\mathbf{3467}, \mathbf{4}5|8|2|1\mathbf{37}9|\mathbf{6}, \mathbf{6}|1|59|2\mathbf{37}|8|\mathbf{4})$ of Example 2.3, we may set

$$z_{\mathbf{6}} = \mathbf{1} < z_1 = z_{\mathbf{3}} = z_{\mathbf{7}} = z_9 = \mathbf{2} < z_2 = 2.8 < z_8 = 2.9 < z_{\mathbf{4}} = z_5 = \mathbf{3}$$

$$w_{\mathbf{4}} = \mathbf{1} < w_8 = 1.9 < w_2 = w_{\mathbf{3}} = w_{\mathbf{7}} = \mathbf{2} < w_5 = w_9 = 2.8 < w_1 = 2.9 < w_{\mathbf{6}} = \mathbf{3}.$$

By construction, the order of the z_is (resp. the w_is) is the opposite of the order dictated by π_1 (resp. π_2). Also $z_k + w_k = c$ is constant for $k \in K$. It remains to show that $z_j + w_j > c$ for $j \notin K$. Assume contrariwise that $z_j + w_j \leq c$. Then for any $k \in K$ we must have $z_j \leq z_k$ or $w_j \leq w_k$. Assume it is the former, and choose $k \in K$ where z_k is minimum such that $z_k \geq z_j$. By construction, we have $z_j > z_k - \epsilon$. Furthermore j comes after k in π_1, so it must come before k in π_2; by construction, we have $w_j > (w_k + 1) - \epsilon$. Thus $z_j + w_j > c + 1 - 2\epsilon > c$, a contradiction. We conclude that τ is the label of a face of the harmonic fan containing (z, w), as desired.

2. The set of points $(z, w) \in \mathsf{N}_n \times \mathsf{N}_n$ that give rise to the ordered set partitions π_1 and π_2 have $(\ell(\pi_1) - 1) + (\ell(\pi_2) - 1)$ degrees of freedom. The condition that $z_k + w_k$ are equal for all $k \in K$ introduces $\ell(\pi_1|K) - 1 = \ell(\pi_2|K) - 1$ linear constraints.

3. To go up the face poset from the face indexed by $(K; \pi_1, \pi_2)$, we need to turn some of the defining equalities into inequalities. The effect of this on the label is to remove elements from K and break a parts of π_1 and π_2 into adjacent parts. $\square$

Using Proposition 2.4, one may check that the harmonic fan is neither simple nor simplicial, already for $n = 3$. We now give the vertex and inequality description of the harmonic polytope.

Proposition 2.5. *The number of vertices of the harmonic polytope $H_{n,n}$ is*

$$v(H_{n,n}) = (n!)^2 \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right).$$

Proof. By Proposition 2.4 we need to count the fine harmonic triples $\tau = (K; \pi_1, \pi_2)$; these are the ones where $K = \{k\}$ and both π_1 and π_2 are permutations. To specify τ, we first specify the element k. Out of the remaining $n-1$ elements, we choose which a of them precede k in π_1 and follow k in π_2, which b of them precede k in both π_1 and π_2, and which c of them follow k in π_1 and precede k in π_2. Finally we choose the order of the $a+b$ elements preceding k in π_1, the order of the c elements following k in π_1, the order of the $b+c$ elements preceding k in π_2, and the order of the a elements following k in π_2. It follows that

$$\begin{aligned}
v(H_{n,n}) &= n \sum_{a+b+c=n-1} \binom{n-1}{a,b,c} (a+b)!\, c!\, a!\, (b+c)! \\
&= n! \sum_{a+b+c=n-1} \frac{(a+b)!(b+c)!}{b!} \\
&= n! \sum_{a=0}^{n-1} \left(a!(n-1-a)! \sum_{b=0}^{n-1-a} \binom{a+b}{a} \right) \\
&= n! \sum_{a=0}^{n-1} \left(a!(n-1-a)! \binom{n}{a+1} \right) \\
&= (n!)^2 \sum_{a=0}^{n-1} \frac{1}{a+1},
\end{aligned}$$

as desired. $\square$

Let us give a concrete description of the vertices of $H_{n,n}$.

Proposition 2.6. *The vertices of the harmonic polytope $H_{n,n}$ are*

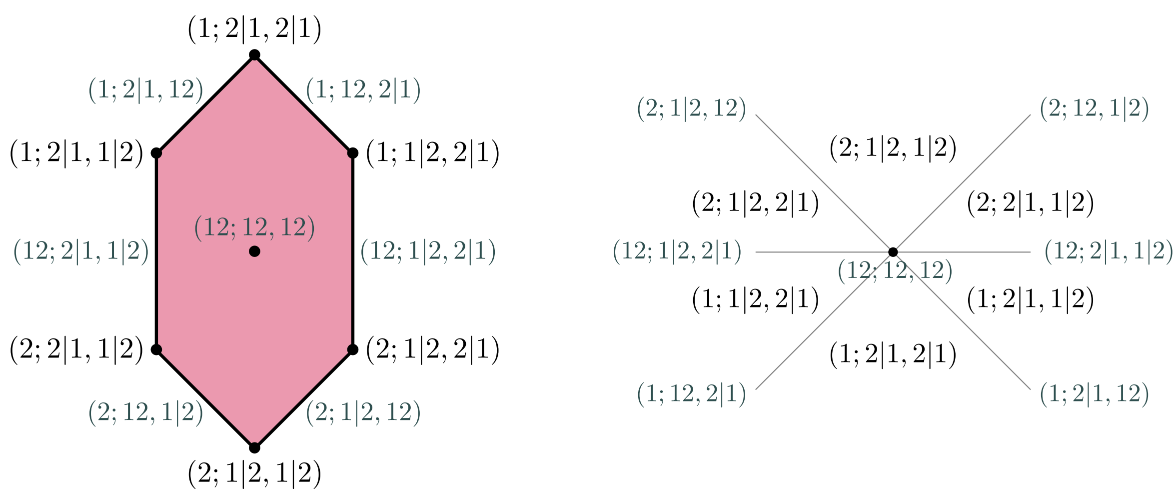
$$v_\tau = \mathsf{e}_k + \mathsf{f}_k + (\pi_1^{-1}, 0) + (0, \pi_2^{-1})$$

for the fine harmonic triples $\tau = (k; \pi_1, \pi_2)$ on $[n]$, where π^{-1} denotes the inverse of the permutation π in one-line notation.

Proof. Consider a point (z, w) in the interior of the chamber of the normal fan $\mathcal{N}(H_{n,n})$ corresponding to a fine harmonic triple $\tau = (k; \pi_1, \pi_2)$. The minimal vertex of $H_{n,n}$ in the direction (z, w) is

$$\begin{aligned}
(H_{n,n})_{z,w} &= (D_n)_{(z,w)} + (\Pi_n \times 0)_{(z,w)} + (0 \times \Pi_n)_{(z,w)} \\
&= (\mathsf{e}_k + \mathsf{f}_k) + (\pi_1^{-1}, 0) + (0, \pi_2^{-1})
\end{aligned}$$

as desired. $\square$



$(k; \pi_1, \pi_2)$	$(2; 1\|2, 1\|2)$	$(2; 1\|2, 2\|1)$	$(2; 2\|1, 1\|2)$	$(1; 1\|2, 2\|1)$	$(1; 2\|1, 1\|2)$	$(1; 2\|1, 2\|1)$
$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$	$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$	$\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$

Figure 1: The harmonic polytope $H_{2,2}$ in $\mathbb{Z}^2 \times \mathbb{Z}^2$ and its reduced normal fan. The faces correspond to the harmonic triples on [2]. The table lists the fine harmonic triples $(k; \pi_1, \pi_2)$ and the corresponding vertices of $H_{2,2}$.

For example, Figure 1 shows how the harmonic polytope $H_{2,2}$ sits in the lattice $\mathbb{Z}^2 \times \mathbb{Z}^2$. Its inner normal fan is the harmonic fan, which coincides with the bipermutohedral fan (only) for $n = 2$; the orientation shown here matches the one in [1, Figure 4].

For a larger example, the vertex of the harmonic polytope $H_{5,5}$ corresponding to the fine harmonic triple $\tau = (4; 53412, 14352)$ on [5] is

$$v_T = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 5 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 3 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 2 & 4 & 1 \\ 1 & 5 & 3 & 3 & 4 \end{pmatrix}$$

since $53412^{-1} = 45231$ and $14352^{-1} = 15324$.

Proposition 2.7. *The number of facets of the harmonic polytope $H_{n,n}$ is $3^n - 3$.*

Proof. In light of Proposition 2.4 we need to enumerate the coarse harmonic triples; that is, those for which $\ell(\pi_1) + \ell(\pi_2) - \ell(\pi_1|K) - 1 = 1$. We consider three cases.

(i) $\ell(\pi_1) = 1$: In this case $\ell(\pi_1|K) = 1$ so we must have $\ell(\pi_2) = 2$; say $\pi_2 = S|T$. Then we have $\tau = (K; [n], S|T)$, and for this triple to be harmonic we must have $K = T$. Therefore $\tau = (T; [n], S|T)$. The corresponding ray of the harmonic fan is $\mathsf{e}_{[n]} + \mathsf{f}_T$.

(ii) $\ell(\pi_2) = 1$: Similarly we obtain $\tau = (T; S|T, [n])$. The corresponding ray of the harmonic fan is $\mathsf{e}_S + \mathsf{f}_{[n]}$.

(iii) $\ell(\pi_1) > 1$ and $\ell(\pi_2) > 1$: Since $(\ell(\pi_1) - \ell(\pi_1|K)) + (\ell(\pi_2) - 2) = 0$ and both summands are nonnegative, we must have $\ell(\pi_2) = 2$ and $\ell(\pi_1) = \ell(\pi_1|K)$. Similarly $\ell(\pi_1) = 2$ and $\ell(\pi_2) = \ell(\pi_2|K)$. Let us write

$$\pi_1 = S|S', \qquad \pi_2 = T|T', \qquad \text{and} \qquad \pi_1|K = K_S|K_{S'}, \qquad \pi_2|K = K_T|K_{T'}.$$

Since $K_S|K_{S'} = K_{T'}|K_T$, an element $j \in S' - K_{S'}$ would contradict Definition 2.2.1(b), so we must have $S' = K_{S'} = K_T$. Similarly $T' = K_{T'} = K_S$. Then $K = K_S \cup K_{S'} = S' \cup T'$. Also $S' \cap T' = K_S \cap K_{S'} = \emptyset$, so $S \cup T = [n]$. Thus

$$\tau = ([n] - (S \cap T);\, S|([n] - S),\, T|([n] - T)) \qquad \text{for } S \cup T = [n].$$

The corresponding ray of the normal fan is $\mathsf{e}_S + \mathsf{f}_T$.

We conclude that the rays of the harmonic fan are the vectors $\mathsf{e}_S + \mathsf{f}_T$ where S and T are non-empty, they are not both equal to $[n]$, and $S \cup T = [n]$. There are $3^n - 3$ such vectors because we can choose freely, for each $i \in [n]$, whether (a) i is in S and not T, (b) i is in T and not S, or (c) i is in both S and T; but the three pairs $([n], \emptyset), (\emptyset, [n])$ and $([n], [n])$ are invalid. $\square$

As introduced in [1], a *bisubset* of $[n]$ is a pair $S|T$ of nonempty subsets of $[n]$, not both equal to $[n]$, such that $S \cup T = [n]$. The previous proof shows that they are in correspondence with the facets of $H_{n,n}$. More precisely, we have:

Proposition 2.8. *The harmonic polytope $H_{n,n}$ is given by the following minimal inequality description:*

$$\begin{aligned} \sum_{e \in [n]} x_e &= \frac{n(n+1)}{2} + 1, \\ \sum_{e \in [n]} y_e &= \frac{n(n+1)}{2} + 1, \\ \sum_{s \in S} x_s + \sum_{t \in T} y_t &\geq \frac{|S|(|S|+1) + |T|(|T|+1)}{2} + 1 \qquad \textit{for each bisubset } S|T \textit{ of } [n]. \end{aligned}$$

Proof. The first two equations hold, and determine a codimension two subspace perpendicular to the lineality space $\mathbb{R}\{\mathsf{e}_{[n]}, \mathsf{f}_{[n]}\}$ of $\mathcal{N}(H_{n,n})$. The minimal inequality description is then determined by the rays $\mathsf{e}_S + \mathsf{f}_T$ for the bisubsets $S|T$. We have

$$\begin{aligned} \min_{(x,y) \in H_{n,n}} (\mathsf{e}_S + \mathsf{f}_T)(x,y) &= \min_{(x,y) \in D_n} (\mathsf{e}_S + \mathsf{f}_T)(x,y) + \min_{(x,y) \in \Pi_n \times 0} \mathsf{e}_S(x,0) + \min_{(x,y) \in 0 \times \Pi_n} \mathsf{f}_T(0,y) \\ &= 1 + (1 + 2 + \cdots + |S|) + (1 + 2 + \cdots + |T|), \end{aligned}$$

which implies the given description. $\square$

We now offer an alternative description of the faces of the harmonic polytope, which gives rise to a formula for the f-vector. A *harmonic table* T on $[n]$ of size $\ell =: \ell(\mathsf{T})$ is a triangular table having $2\ell + 1$ rows and $2\ell + 1$ columns of lengths $2\ell + 1, 2\ell - 1, 2\ell - 1, \ldots, 3, 3, 1, 1$, respectively, decorated with the following data:

- A labeling of the even columns with nonempty, pairwise disjoint subsets of $[n]$.
- A labeling of the even rows with the same subsets, listed in the opposite order.
- A placement of each element of $[n]$ not used as a row or column label in one box of the table.

We let $c_i(\mathsf{T})$ and $r_i(\mathsf{T})$ denote the number of elements in column $2i + 1$ and row $2i + 1$ of T, respectively. Figure 2 shows a harmonic table of size 3 on $[9]$, with $c_1(\mathsf{T}) = 2$, $r_1(\mathsf{T}) = 3$, $r_2(\mathsf{T}) = 1$, and all other $c_i(\mathsf{T})$ and $r_i(\mathsf{T})$ equal to 0.

Let $F(m)$ be the mth *Fubini number* (or *ordered Bell number*), which counts the ordered set partitions of $[m]$. Also recall that the *Stirling number of the second kind* $S(m, p)$ counts the number of unordered set partitions of m into p parts.

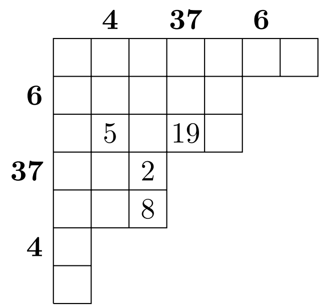


Figure 2: A harmonic table of size 3 on [9].

Proposition 2.9. *The number of faces of the harmonic polytope $H_{n,n}$ is*

$$f(H_{n,n}) = \sum_{\mathsf{T}} \prod_{i=0}^{\ell(\mathsf{T})} \Big(F(c_i(\mathsf{T}))F(r_i(\mathsf{T})) \Big)$$

summing over all harmonic tables on $[n]$. The number of d-dimensional faces of $H_{n,n}$ is

$$f_d(H_{n,n}) = \sum_{\mathsf{T}} \sum_{\mathsf{a},\mathsf{b}} \prod_{i=0}^{\ell(\mathsf{T})} \Big(S(c_i(\mathsf{T}), a_i)\, a_i!\, S(r_i(\mathsf{T}), b_i)\, b_i! \Big)$$

summing over all harmonic tables T on $[n]$ and all sequences $\mathsf{a} = (a_0, a_1, \ldots, a_\ell)$ and $\mathsf{b} = (b_0, b_1, \ldots, b_\ell)$ with $\ell = \ell(\mathsf{T})$ and $\sum_i a_i + \sum_i b_i + \ell = 2n - d - 1$.

Proof. A harmonic triple $T = (K; \pi_1, \pi_2)$ can be constructed in five steps, with the help of a harmonic table, as follows. This process is illustrated in Figure 2, which shows the harmonic table that gives rise to the harmonic triple $T = (\mathbf{3467}, \mathbf{4}5|8|2|1\mathbf{379}|\mathbf{6}, \mathbf{6}|1|59|2\mathbf{37}|8|\mathbf{4})$ of Example 2.3.

1. Choose the subset K of $[n]$.

2. Choose the ordered set partition $\pi_1|K =: K_1|\cdots|K_\ell$. This automatically determines $\pi_2|K$, which is its reverse. Record this data on a triangular table T of size ℓ, labeling the $2i$th column from left to right and the $2i$th row from bottom to top with the set K_i.

3. Choose, for each element $j \in J := [n] - K$, its position relative to $K_1, \ldots, K_\ell$ in the ordered set partition π_1, and its position relative to $K_1, \ldots, K_\ell$ in the ordered set partition π_2. Record this information in the table T as follows. If j is in the same block as K_i in π_1 (resp. in π_2), put it in the column (resp. row) labeled by K_i in T. If j is between blocks K_i and K_{i+1} in π_1 (resp. in π_2), put it in the unlabeled column (rep. row) between the columns (resp. rows) labeled by K_i and K_{i+1} in T. Notice that, by item 1(b) in the Definition 2.2 of a harmonic triple, all these numbers will land inside the triangular table.

4. Choose the relative order of the elements of $J = [n] - K$ in the ordered set partition π_1. To do this, it suffices to choose, for each i, the relative order of the elements of J that appear between blocks K_i and K_{i+1} of π_1 for each i. These are precisely the $c_i(\mathsf{T})$ elements in column $2i+1$, and their order is given by an arbitrary ordered set partition of that size, so there are $F(c_1(\mathsf{T})) \cdots F(c_\ell(\mathsf{T}))$ such choices.

5. Choose the relative order of the elements of $J = [n] - K$ in the ordered set partition π_2. As in step 4, there are $F(r_1(\mathsf{T})) \cdots F(r_\ell(\mathsf{T}))$ such choices.

Each harmonic triple on $[n]$ – and hence each face of the harmonic polytope $H_{n,n}$ – arises in a unique way from this procedure. This proves the first formula.

By Proposition 2.4.2, the d-dimensional faces of the harmonic polytope $H_{n,n}$ dimension d correspond to the harmonic triples $(\tau; \pi_1, \pi_2)$ with $d = \ell(\pi) + \ell(\pi_2) - \ell(\pi_1|K) - 1$. For the harmonic table T given by $(\tau; \pi_1, \pi_2)$, let a_i (resp. b_i) denote the length of the ordered set partition of the $c_i(\mathsf{T})$ elements in column $2i+1$ (resp. the $r_i(\mathsf{T})$ elements in column $2i+1$) has length a_i (resp. b_i), for $i = 1, \ldots, \ell(\mathsf{T})$. Then $\ell(\pi_1) = \ell(\mathsf{T}) + \sum_i a_i$ and $\ell(\pi_2) = \ell(\mathsf{T}) + \sum_i b_i$, and $\ell(\pi_1|K) = \ell(\mathsf{T})$, so $d = \ell(\mathsf{T}) + \sum_i a_i + \sum_i b_i - 1$. There are $S(c_i(\mathsf{T}), a_i)\, a_i!$ (resp. $S(r_i(\mathsf{T}), b_i)\, b_i!$) such ordered set partitions for each i, from which the result follows. $\square$

For fixed k and ℓ, there are $\binom{n}{k}$ choices for a set K of k elements, there are $\ell!\, S(k, \ell)$ choices for an ordered set partition $K_1|\cdots|K_\ell$ of K of size ℓ, and there are $(2\ell+1)+2(2\ell-1)+\cdots+2(3)+2(1) = 2\ell^2+2\ell+1$ choices for where to place each element not in K in the harmonic table. Therefore the number of harmonic tables for $[n]$ is

$$\sum_{k=1}^{n-1} \sum_{\ell=1}^{k} \binom{n}{k} S(k, \ell)\, \ell!\, (2\ell^2 + 2\ell + 1)^{n-k}.$$

Using Proposition 2.9 one can compute the f-vector of the first few harmonic polytopes:

$$\begin{aligned}
f(H_{1,1}) &= (1, 1), \\
f(H_{2,2}) &= (1, 6, 6, 1), \\
f(H_{3,3}) &= (1, 66, 144, 102, 24, 1), \\
f(H_{4,4}) &= (1, 1200, 4008, 5124, 3072, 834, 78, 1).
\end{aligned}$$

2.2 The harmonic polytope and the bipermutohedron

As stated in the introduction, the harmonic polytope is one of two polytopes that arose in Ardila, Denham, and Huh's work on the Lagrangian geometry of matroids. The other one is the *bipermutohedron*. We now describe the combinatorial relationship between them. Only for this subsection, we assume familiarity with the construction of the bipermutohedral fan $\Sigma_{n,n}$ and the bipermutahedron $\Pi_{n,n}$ in [1, Section 2].

We have shown that the harmonic polytope has $3^n - 3$ facets and $(n!)^2(1 + \frac{1}{2} + \cdots + \frac{1}{n})$ vertices. In turn, the bipermutohedron has $3^n - 3$ facets and $(2n)!/2^n$ vertices. The harmonic polytope is a Minkowski summand of (a multiple of) the bipermutohedron, as shown by the following proposition, originally discovered in [1]. We give an alternative proof that makes the combinatorial relationship between these objects more explicit.

Proposition 2.10. *The harmonic fan is a coarsening of the bipermutohedral fan.*

Proof. Suppose a point $(z, w) \in \mathsf{N}_n \times \mathsf{N}_n$ is in the interior of cone σ_{B} of the bipermutohedral fan $\Sigma_{n,n}$, corresponding to a bisequence B. Then $z_k + w_k$ is minimized precisely for the set K of indices $k \in [n]$ that appear only once in B. This places the point (z, w) in the chart $\mathscr{C}_k$ of the bipermutohedral fan for each $k \in K$. Fix one such k.

Now, as explained in [1, Proposition 2.9], the order of the first occurrences of each $i \in [n]$ in the bisequence B is determined by the reverse order of the numbers $Z_i = z_i - z_k$, which is the reverse order of the z_is. Similarly, the order of the second occurrences of each $i \in [n]$ in the bisequence B is determined by the reverse order of the numbers $W_i = w_k - w_i$, which is the order of the w_is.

We conclude that (z, w) is in the interior of the cone of the harmonic fan indexed by the harmonic triple $(K; \pi_1, \pi_2)$ where K is the set of elements of $[n]$ appearing only once in B, π_1 is the ordered set partition obtained from the order of the first occurrence of each i in B, and π_2 is the ordered set partition obtained by reversing the order of the second occurrence of each i in B. $\square$

For example, if (z, w) is in the interior of the cone of the bipermutohedral fan $\Sigma_{6,6}$ indexed by the bisequence $\mathsf{B} = 34|2|356|1|247|6$, then (z, w) is in the interior of the cone of the harmonic fan $\mathcal{N}(H_{6,6})$ indexed by the harmonic triple $\tau = (\mathbf{157};\ 34|2|\mathbf{56}|\mathbf{1}|\mathbf{7},\ 6|24\mathbf{7}|\mathbf{1}|3\mathbf{5})$.

3 The volume of $H_{n,n}$

The goal of this section is to compute the volume of the harmonic polytope. We do so using the theory of mixed volumes, and the Bernstein-Khovanskii-Kushnirenko Theorem which relates these volumes to the enumeration of solutions of systems of polynomial equations. Before reviewing the basics of this theory, let us comment on the definition of volume used here.

Normalizing the volume. Most of the polytopes that we study are not full-dimensional in their ambient space, and we need to define their volumes and mixed volumes carefully. Let P be a d-dimensional polytope on an affine d-plane $L \subset \mathbb{Z}^n$. Assume $L \cap \mathbb{Z}^n$ is a lattice translate of a d-dimensional lattice Λ. We call a lattice d-parallelotope in L *primitive* if its edges generate the lattice Λ; all primitive parallelotopes have the same volume. Then we define the *normalized volume* of a d-polytope P in L to be $\mathsf{Vol}(P) := \mathsf{EVol}(P)/\mathsf{EVol}(\square)$ for any primitive parallelotope $\square$ in L, where EVol denotes Euclidean volume. By convention, the normalized volume of a point is 1. Throughout the paper, all volumes and mixed volumes are normalized in this way.

3.1 The volume of the harmonic polytope in terms of mixed volumes

Theorem 3.1. *(McMullen, [12]) There is a unique function $\mathsf{MV}(Q_1, \ldots, Q_d)$ defined on d-tuples of polytopes in $\mathbb{R}^d$, called the* mixed volume *such that for any collection of polytopes $P_1, \ldots, P_m$ in $\mathbb{R}^d$ and any nonnegative real numbers $\lambda_1, \ldots, \lambda_m$, we have*

$$\mathsf{Vol}(\lambda_1 P_1 + \cdots + \lambda_m P_m) = \sum_{i_1, \ldots, i_d} \mathsf{MV}(P_{i_1}, \ldots, P_{i_d}) \lambda_{i_1} \cdots \lambda_{i_d}, \tag{1}$$

summing over all ordered d-tuples $(i_1, \ldots, i_d)$ with $1 \leq i_k \leq m$ for $1 \leq k \leq d$. Moreover, the function $\mathsf{MV}(Q_1, \ldots, Q_d)$ is symmetric; that is, $\mathsf{MV}(Q_1, \ldots, Q_d) = \mathsf{MV}(Q_{\sigma(1)}, \ldots, Q_{\sigma(d)})$ for any permutation σ of $[d]$.

Mixed volumes have the following algebraic interpretation.

Theorem 3.2 (Bernstein-Khovanskii-Kushnirenko Theorem)**.** *[4] Let $A_1, \dots, A_d \subset \mathbb{Z}^d$ be d finite sets of lattice points, and let $Q_i = \mathsf{conv}(A_i)$ for $i = 1, \dots, d$. If the number of solutions in the torus $(\mathbb{C}^*)^d$ to the system*

$$\begin{cases} \displaystyle\sum_{\alpha \in A_1} \lambda_{1,\alpha} x^\alpha = 0, \\ \qquad\qquad \vdots \\ \displaystyle\sum_{\alpha \in A_d} \lambda_{d,\alpha} x^\alpha = 0 \end{cases}$$

is finite for a given choice of complex coefficients $\lambda_{i,\alpha}$, then that number is bounded above by $d!\,\mathsf{MV}(Q_1, \dots, Q_d)$. Moreover, if the coefficients $\lambda_{i,\alpha}$ are sufficiently generic, the number of solutions equals $d!\,\mathsf{MV}(Q_1, \dots, Q_d)$.

The BKK Theorem is most often used to count or bound the solutions to a system of polynomial equations by computing the corresponding mixed volume. As Postnikov points out in [9], it can also be used in the reverse direction. This will be our approach: we will compute mixed volumes by counting the solutions to the associated systems of polynomial equations. This is seldom possible; but in our case, the resulting enumeration problems can be expressed in terms of the combinatorics of toric ideals of graphs and the enumeration of lattice points in polytopes.

To apply this general discussion to the harmonic polytope, we begin by defining the segments

$$\Delta_{ij} := \mathsf{conv}\{\mathsf{e}_i, \mathsf{e}_j\} \qquad \text{and} \qquad \Delta_{\bar{i}\bar{j}} := \mathsf{conv}\{\mathsf{f}_i, \mathsf{f}_j\} \qquad \text{for } 1 \leq i < j \leq n.$$

The permutohedron equals $\Pi_n = \mathsf{e}_{[n]} + \sum_{i<j} \Delta_{ij}$ [16] so the harmonic polytope is the Minkowski sum

$$H_{n,n} = \mathsf{e}_{[n]} + \mathsf{f}_{[n]} + \sum_{i<j} \Delta_{ij} + \sum_{i<j} \Delta_{\bar{i}\bar{j}} + D_n \quad \subset \quad \mathbb{R}^n \times \mathbb{R}^n. \tag{2}$$

The first two summands $\mathsf{e}_{[n]}$ and $\mathsf{f}_{[n]}$ are simply translations, so we focus on the remaining ones. Given graphs G and G' on vertex set $[n]$ with edge multisets $\{i_1j_1, \dots, i_rj_r\}$ and $\{\bar{i}_1\bar{j}_1, \dots, \bar{i}_s\bar{j}_s\}$, respectively, we define their mixed volume to be

$$\mathsf{MV}(G, G') := \mathsf{MV}(\Delta_{i_1j_1}, \dots, \Delta_{i_rj_r}, \Delta_{\bar{i}_1\bar{j}_1}, \dots, \Delta_{\bar{i}_s\bar{j}_s}, \underbrace{D_n, \dots, D_n}_{k \text{ times}}) \tag{3}$$

where $k = 2n - 2 - r - s$. We also let $\binom{2n-2}{G,G';D_n}$ denote the number of distinct permutations of the sequence $(\Delta_{i_1j_1}, \dots, \Delta_{i_rj_r}, \Delta_{\bar{i}_1\bar{j}_1}, \dots, \Delta_{\bar{i}_s\bar{j}_s}, D_n, \dots, D_n)$. Combining (1) with the fact that mixed volumes are symmetric, we obtain:

$$\mathsf{Vol}(H_{n,n}) = \sum_{G,G'} \binom{2n-2}{G, G'; D_n} \mathsf{MV}(G, G'), \tag{4}$$

summing over all pairs of graphs G and G' on $[n]$. Therefore it remains to compute the mixed volumes $\mathsf{MV}(G, G')$.

Remark 3.3. The normalized volume $\mathsf{Vol}(H_{n,n})$ is equal to the Euclidean volume of the projection of $H_{n,n}$ onto $\mathbb{Z}^{\{2,\dots,n\}}\times\mathbb{Z}^{\{2,\dots,n\}}$. This projection of $H_{n,n}$ is equal to the Minkowski sum of the images under this projection of the polytopes appearing in (2). Thus $\mathsf{MV}(G,G')$ is the mixed volume of the projections of $\Delta_{i_1j_1},\dots,\Delta_{i_rj_r},\Delta_{\bar{i}_1\bar{j}_1},\dots,\Delta_{\bar{i}_s\bar{j}_s},D_n,\dots,D_n$ onto $\mathbb{Z}^{\{2,\dots,n\}}\times\mathbb{Z}^{\{2,\dots,n\}}$.

The BKK Theorem then tells us that $(2n-2)!\,\mathsf{MV}(G,G')$ counts the solutions in $(\mathbb{C}^*)^n\times(\mathbb{C}^*)^n$ to the following system of equations:

$$\mathcal{E}(G,G'):\quad \begin{cases} x_i=\lambda_{ij}x_j, \text{ for } ij\in E(G) & \nu_{11}x_1y_1+\cdots+\nu_{1n}x_ny_n=0 \\ y_i=\mu_{ij}y_j, \text{ for } \bar{i}\bar{j}\in E(G') & \vdots \\ x_1=y_1=1 & \nu_{k1}x_1y_1+\cdots+\nu_{kn}x_ny_n=0, \end{cases} \tag{5}$$

where G and G' have r and s edges respectively, $k:=2n-2-r-s$, and the coefficients $\lambda_{ij},\mu_{ij},\nu_{ij}$ are chosen generically.

3.2 Mixed volumes

In this section we compute the mixed volumes (3) of the harmonic polytope. We begin by showing that most of them vanish.

Lemma 3.4. *If G or G' contains a cycle then the mixed volume* $\mathsf{MV}(G,G')=0$.

Proof. By Theorem 3.2, $(2n-2)!\,\mathsf{MV}(G,G')$ counts the solutions in $(\mathbb{C}^*)^n\times(\mathbb{C}^*)^n$ to the system of equations $\mathcal{E}(G,G')$. Suppose that G contains the cycle

$$i_1\to i_2\to\cdots\to i_\ell\to i_1.$$

for some vertices $i_1,\dots,i_\ell\in[n]$. The equations of the corresponding k edges

$$x_{i_1}=\lambda_{i_1i_2}x_{i_2}\qquad\dots\qquad x_{i_{k-1}}=\lambda_{i_{k-1}i_k}x_{i_k}\qquad x_{i_k}=\lambda_{i_ki_1}x_{i_1}$$

imply that $x_{i_1}=(\lambda_{i_1i_2}\cdots\lambda_{i_{k-1}i_k}\lambda_{i_ki_1})x_{i_1}$. Since the λ_{ij}s are chosen generically, the only solution to this equation is $x_{i_1}=0$. It follows that the system of equations $\mathcal{E}(G,G')$ has no solutions in the torus $(\mathbb{C}^*)^n\times(\mathbb{C}^*)^n$, and $\mathsf{MV}(G,G')=0$. $\square$

Our next goal is to describe the non-zero mixed volumes $\mathsf{MV}(G,G')$. To accomplish it, we will require some additional constructions.

The Bipartite Graph and the Root Polytope. Fix graphs G and G' on $[n]$. Let $\mathcal{I}=\{I_1,\dots,I_p\}$ and $\mathcal{J}=\{J_1,\dots,J_q\}$ be the set partitions of $[n]$ into connected components of G and G', respectively. Let $I(k)$ and $J(k)$ denote the parts of $\mathcal{I}$ and $\mathcal{J}$ containing vertex k for $k\in[n]$. Define the bipartite graph $\Gamma=\Gamma_{\mathcal{I},\mathcal{J}}$ with vertex set $\mathcal{I}\cup\mathcal{J}$ and n edges $I(k)J(k)$ for $1\le k\le n$. This graph may have several edges connecting the same pair of vertices. We give the edge $I(k)J(k)$ the label k. Notice that the label of a vertex in Γ is just the set of labels of the edges containing it. Therefore we can remove the vertex labels, and simply think of Γ as a bipartite multigraph on edge set $[n]$.

The *edge polytope* of Γ is

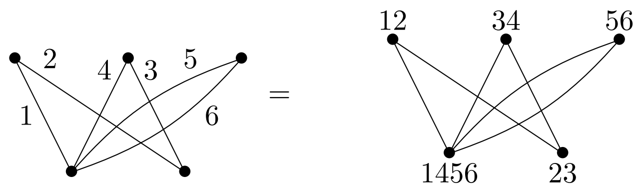
$$\begin{aligned} R_\Gamma &:= \mathsf{conv}\{\mathsf{e}_{I_a}+\mathsf{f}_{J_b}\,:\,I_a\in\mathcal{I},\,J_b\in\mathcal{J},\,I_a\cap J_b\neq\varnothing\} \\ &= \mathsf{conv}\{\mathsf{e}_{I(1)}+\mathsf{f}_{J(1)},\dots,\mathsf{e}_{I(n)}+\mathsf{f}_{J(n)}\}\subset\mathbb{R}^p\times\mathbb{R}^q, \end{aligned}$$

writing $\mathsf{e}_{I(1)},\dots,\mathsf{e}_{I(p)}$ and $\mathsf{f}_{J(1)},\dots,\mathsf{f}_{J(p)}$ for the standard bases of $\mathbb{R}^p\cong\mathbb{R}^{\mathcal{I}}$ and $\mathbb{R}^q\cong\mathbb{R}^{\mathcal{J}}$. This polytope lives on the codimension 2 subspace cut out by the equations $x_1+\cdots+x_p=y_1+\cdots+y_q=1$.

Example 3.5. Consider the following graphs

The partitions corresponding to these graphs are $\mathcal{I} = \{12, 34, 56\}$ and $\mathcal{J} = \{1456, 23\}$, omitting brackets for easier readability. The associated bipartite multigraph is

$\Gamma_{\mathcal{I},\mathcal{J}} =$



and the corresponding edge polytope is

$$R_\Gamma = \mathsf{conv}(\mathsf{e}_a + \mathsf{f}_A,\ \mathsf{e}_a + \mathsf{f}_B,\ \mathsf{e}_b + \mathsf{f}_B,\ \mathsf{e}_b + \mathsf{f}_A,\ \mathsf{e}_c + \mathsf{f}_A,\ \mathsf{e}_c + \mathsf{f}_A) \subset \mathbb{R}^{abc} \times \mathbb{R}^{AB},$$

writing $a = 12, b = 34, c = 56$ and $A = 1456, B = 23$.

Lemma 3.6. *The only lattice points of the edge polytope R_Γ are its vertices.*

Proof. The polytope R_Γ is contained in the sphere S centered at the origin with radius $\sqrt{2}$, so it can only contain lattice points of norm $0, 1$, or $\sqrt{2}$. Since R_Γ lies on the hyperplanes $\sum_i x_i = 1$ and $\sum_i y_i = 1$, it cannot contain a lattice point of norm 0 (the origin) or 1 (a point of the form $\pm\mathsf{e}_i$ or $\pm\mathsf{f}_i$). Therefore every lattice point in R_Γ must be of the form $\mathsf{e}_i + \mathsf{f}_j$ for some $i, j \in [n]$. These points are all on the surface of the sphere S, so they are in convex position; therefore, if a point $\mathsf{e}_i + \mathsf{f}_j$ is in R_Γ, it must in fact be a vertex of R_Γ. The result follows. $\square$

A lattice polytope P is *normal* if for all positive integers k and all lattice points x in kP there exist lattice points $x_1, \ldots, x_k$ in P such that $x = x_1 + \cdots + x_k$. A lattice polytope P is *very ample* if the above property holds for all sufficiently large integers k. This is a favorable property algebro-geometrically, because if P is very ample then the lattice points of P provide a concrete projective embedding of the toric variety X_P of P, as follows. Let $P \cap \mathbb{Z}^d = \{\mathsf{a}_1, \ldots, \mathsf{a}_s\} =: A$. The projective embedding of X_P is the Zariski closure of the image of the map

$$\begin{array}{rcl} (\mathbb{C}^*)^d & \longrightarrow & \mathbb{CP}^{s-1} \\ \mathbf{t} & \longmapsto & (\mathbf{t}^{\mathsf{a}_1}, \ldots, \mathbf{t}^{\mathsf{a}_s}) \end{array}$$

and its defining ideal I_A is the kernel of the homomorphism

$$\begin{array}{rcl} \varphi : \mathbb{C}[x_1, \ldots, x_s] & \longrightarrow & \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}] \\ x_i & \longmapsto & \mathbf{t}^{\mathsf{a}_i}. \end{array}$$

The map above induces the map of lattices

$$\begin{aligned} \widehat{\varphi} : \mathbb{Z}^s &\longrightarrow \mathbb{Z}^d \\ \mathsf{e}_i &\longmapsto \mathsf{a}_i, \end{aligned}$$

where $\mathsf{e}_1, \ldots, \mathsf{e}_s$ is the standard basis of $\mathbb{Z}^s$. The kernel of φ is the *toric ideal*

$$I_A = \langle \mathbf{x}^{\mathsf{u}} - \mathbf{x}^{\mathsf{v}} \ : \ \mathsf{u}, \mathsf{v} \in \mathbb{N}^s,\ \widehat{\varphi}(\mathsf{u}) = \widehat{\varphi}(\mathsf{v}) \rangle \ \subset \ \mathbb{C}[x_1, \ldots, x_s];$$

see [5, §2.1 and §2.3].

Proposition 3.7. *If Γ is bipartite, the edge polytope R_Γ is normal.*

Proof. Let

$$C_\Gamma = \mathsf{cone}(R_\Gamma) = \{\lambda\, q \ : \ q \in R_\Gamma, \lambda \geq 0\} \subset \mathbb{R}^n \times \mathbb{R}^n$$

be the cone over the polytope R_Γ. Consider a lattice point x in $k\, R_\Gamma$. The cone C_Γ is generated by the vertices of R_Γ, so x is a positive combination of them. By Caratheodory's theorem, x can be expressed a positive combination of only e linearly independent vertices of R_Γ, say $v_1, \ldots, v_e$, for some $e \leq \dim R_\Gamma$. But the vector configuration $\{\mathsf{e}_i + \mathsf{f}_j \ : \ 1 \leq i, j \leq n\}$ is unimodular, so $v_1, \ldots, v_e$ form a lattice basis for $\mathsf{cone}(v_1, \ldots, v_e) \cap (\mathbb{Z}^n \times \mathbb{Z}^n)$. It follows that x is a positive **integer** combination of $v_1, \ldots, v_e \in R_\Gamma$. We conclude that R_Γ is normal as desired. □

The Toric Ideal, the Toric Variety, and the Trimmed Generalized Permutahedra. The graph $\Gamma = \Gamma_{\mathcal{I},\mathcal{J}}$ gives rise to a ring homomorphism

$$\begin{aligned} \mathbb{R}[z_e \ : e \text{ edge of } \Gamma] &\longrightarrow \mathbb{R}[y_v \ : v \text{ vertex of } \Gamma] \\ z_e &\longmapsto y_i y_j \quad \text{where edge } e \text{ joins vertices } i \text{ and } j \end{aligned}$$

The kernel of this homomorphism is called the *toric ideal* I_Γ of Γ; it is a homogeneous ideal given by the cycles of even length in Γ:

$$I_\Gamma = \langle z_{e_1} z_{e_3} \cdots z_{e_{2k-1}} - z_{e_2} z_{e_4} \cdots z_{e_{2k}} \ : \ e_1 e_2 \cdots e_{2k} \text{ is a cycle of } \Gamma \rangle;$$

see [7, Section 5.3]. This ideal is related to the edge polytope as follows.

Proposition 3.8. *If Γ is a bipartite graph, the projective variety of the toric ideal I_Γ is an embedding of the toric variety X_Γ of the edge polytope R_Γ.*

Proof. This holds thanks to Lemma 3.6 and 3.7; see [5, §2.3]. □

The following polytopes will also play an important role. Consider the Minkowski sums

$$P_\Gamma := \sum_{i=1}^{p} \Delta_{\mathsf{nbr}(I_i)} \subset \mathbb{R}^q \qquad \text{and} \qquad Q_\Gamma := \sum_{j=1}^{q} \Delta_{\mathsf{nbr}(J_j)} \subset \mathbb{R}^p$$

where $\Delta_I := \mathsf{conv}\{\mathsf{e}_i \ : \ i \in I\}$, and where $\mathsf{nbr}(I_i) = \{j \in [q] \ : \ I_i J_j \text{ is an edge of } \Gamma\}$ and $\mathsf{nbr}(J_j) = \{i \in [p] \ : \ I_i J_j \text{ is an edge of } \Gamma\}$ denote the neighborhoods of I_i and J_j in Γ. Finally, define the *trimmed generalized permutahedra* of Γ to be the Minkowski differences

$$P_\Gamma^- := P_\Gamma - \Delta_{[q]} \subset \mathbb{R}^q \qquad \text{and} \qquad Q_\Gamma^- := Q_\Gamma - \Delta_{[p]} \subset \mathbb{R}^p$$

Example 3.9. We return to Example 3.5. The toric ideal of Γ is

$$I_\Gamma = \langle z_1z_3 - z_2z_4, z_5 - z_6 \rangle \subset \mathbb{C}[z_1, z_2, z_3, z_4, z_5, z_6].$$

The generalized permutahedra associated to Γ are

$$P_\Gamma = \Delta_{abc} + \Delta_{ab} \subset \mathbb{R}^{abc} \qquad \text{and} \qquad Q_\Gamma = 2\Delta_{AB} + \Delta_A \subset \mathbb{R}^{AB}$$

and the trimmed generalized permutahedra are

$$P_\Gamma^- = \Delta_{ab} \subset \mathbb{R}^{abc} \qquad \text{and} \qquad Q_\Gamma^- = \Delta_{AB} + \Delta_A \subset \mathbb{R}^{AB}.$$

In general, the polytopes P_Γ^- and Q_Γ^- live in different dimensions and can be very different from each other. However, we will see that they always have the same number of lattice points.

Putting it all Together. We now have all the ingredients to describe the mixed volumes $\mathsf{MV}(G, G')$.

Proposition 3.10. *Let G and G' be acyclic graphs on $[n]$ and Γ be the corresponding bipartite graph, having p and q vertices on each side of the bipartition. The following numbers are equal:*

1. *The $(2n-2)$-dimensional mixed volume $\mathsf{MV}(G, G')$ multiplied by $(2n-2)!$.*
2. *The $(p+q-2)$-dimensional volume of the edge polytope R_Γ multiplied by $(p+q-2)!$.*
3. *The number $i(P_\Gamma^-)$ of lattice points in the trimmed generalized permutahedron P_Γ^- in $\mathbb{R}^q$.*
4. *The number $i(Q_\Gamma^-)$ of lattice points in the trimmed generalized permutahedron Q_Γ^- in $\mathbb{R}^p$.*

Furthermore, the numbers above are zero if and only if Γ is disconnected. If Γ is connected, the numbers above are equal to:

5. *the degree of the projective embedding $V(I_\Gamma)$ of the toric variety X_Γ.*

Recall that all volumes are normalized so the volume of a primitive parallelotope in any dimension is 1.

Proof. Let $\mathcal{E}(G, G')$ be the system of equations (5) associated to the mixed volume $\mathsf{MV}(G, G')$. By Theorem 3.2, the quantity in 1. counts the solutions in $(\mathbb{C}^*)^n \times (\mathbb{C}^*)^n$ to $\mathcal{E}(G, G')$.

(1. = 2.) The set of solutions to $\mathcal{E}(G, G')$ is the variety $V(I_{G,G'} + J)$, where

$$\begin{aligned} I_{G,G'} &:= \big\langle x_i - \lambda_{ij}x_j \,:\, i<j,\, ij \in E(G) \big\rangle + \big\langle y_i - \mu_{ij}y_j \,:\, i<j,\, ij \in E(G') \big\rangle \\ J &:= \big\langle \nu_{i1}x_1y_1 + \cdots + \nu_{in}x_ny_n \,:\, 1 \le i \le k \big\rangle + \langle x_1 - 1, y_1 - 1 \rangle \end{aligned}$$

in $\mathbb{C}[x_1^\pm, \ldots, x_n^\pm, y_1^\pm, \ldots, y_n^\pm]$.

Consider the subspace

$$\mathcal{L} = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \,:\, x_i = \lambda_{ij}x_j \text{ for } ij \in E(G)\,,\, y_i = \mu_{ij}y_j \text{ for } ij \in E(G')\} \subset \mathbb{C}^n \times \mathbb{C}^n$$

and the projection

$$\begin{array}{rcl}\psi^a : \mathcal{L} & \longrightarrow & \mathbb{C}^p \times \mathbb{C}^q \\ (x_1, \ldots, x_n, y_1, \ldots, y_n) & \longmapsto & (x_{\min I_1}, \ldots, x_{\min I_p}, y_{\min J_1}, \ldots, y_{\min J_q}).\end{array}$$

Note that for $(x, y) \in \mathcal{L}$ if we have $x_{\min I_a} = 0$ then $x_i = 0$ for all $i \in I_a$, since I_a is connected in G; the same holds for the ys. Therefore ψ^a is injective and, since $\mathsf{dim}(\mathcal{L}) = p + q = \mathsf{dim}(\mathbb{C}^p \times \mathbb{C}^q)$, it follows that ψ^a is an isomorphism. Moreover, $x_{\min I_a} = 0$ if and only if $x_i = 0$ for some $i \in I_a$; the same holds for the ys. This implies that the restriction of ψ^a to $\mathcal{L} \cap \big((\mathbb{C}^*)^n \times (\mathbb{C}^*)^n\big)$ is a morphism with image $(\mathbb{C}^*)^p \times (\mathbb{C}^*)^q$. This morphism defines the following isomorphism between the coordinate rings of $\mathcal{L} \cap \big((\mathbb{C}^*)^n \times (\mathbb{C}^*)^n\big)$ and $(\mathbb{C}^*)^p \times (\mathbb{C}^*)^q$:

$$\begin{array}{rcl}\psi : \mathbb{C}[x_{I_1}^{\pm}, \ldots, x_{I_p}^{\pm}, y_{J_1}^{\pm}, \ldots, y_{J_q}^{\pm}] & \longrightarrow & \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}, y_1^{\pm}, \ldots, y_n^{\pm}]/I_{G,G'} \\ x_{I_a} & \longmapsto & \bar{x}_{\min I_a} \\ y_{J_b} & \longmapsto & \bar{y}_{\min J_b}.\end{array}$$

Let $\overline{J}$ be the image of J in the quotient $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}, y_1^{\pm}, \ldots, y_n^{\pm}]/I_{G,G'}$. By Noether's isomorphism theorems we have

$$\begin{aligned}\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}, y_1^{\pm}, \ldots, y_n^{\pm}]/(I_{G,G'} + J) &\cong \big(\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}, y_1^{\pm}, \ldots, y_n^{\pm}]/I_{G,G'}\big) / \big((I_{G,G'} + J)/I_{G,G'}\big) \\ &\cong \mathbb{C}[x_{I_1}^{\pm}, \ldots, x_{I_p}^{\pm}, y_{J_1}^{\pm}, \ldots, y_{J_q}^{\pm}]/\psi^{-1}\big((I_{G,G'} + J)/I_{G,G'}\big) \\ &= \mathbb{C}[x_{I_1}^{\pm}, \ldots, x_{I_p}^{\pm}, y_{J_1}^{\pm}, \ldots, y_{J_q}^{\pm}]/\psi^{-1}(\overline{J}). \end{aligned} \tag{6}$$

Note that for $1 \leq m \leq n$ we have $\bar{x}_m = \lambda\, \bar{x}_{\min I(m)}$ and $\bar{y}_m = \mu\, \bar{y}_{\min J(m)}$ for nonzero scalars λ and μ. Thus we have, for $i \in [k]$,

$$\psi^{-1}(\nu_{i1}\bar{x}_1\bar{y}_1 + \cdots + \nu_{in}\bar{x}_n\bar{y}_n) = \eta_{i1}x_{I(1)}y_{J(1)} + \cdots + \eta_{in}x_{I(n)}y_{J(n)}$$

for some nonzero constants η_{ij} that are generic if the $\nu_{ij}s$ are sufficiently generic. We conclude that $\psi^{-1}(\overline{J})$ is generated by k generic equations whose Newton polytope is equal to R_Γ, together with $x_{I(1)} - 1$ and $y_{J(1)} - 1$. Recall that $k = 2n - 2 - r - s$ where r and s are the numbers of edges of G and G' respectively. Since these graphs are acyclic, $r = n - p$ and $s = n - q$, so $k = p + q - 2$ equals the ambient dimension of the Newton polytope R_Γ.

By the BKK Theorem, the left-hand side of (6) is a variety consisting of $(2n-2)!\,\mathsf{MV}(G, G')$ points and the right hand side is a variety consisting of $(p+q-2)!\,\mathsf{Vol}_{p+q-2}(R_\Gamma)$ points. Therefore these two numbers are equal to each other.

(2. $=$ 3. $=$ 4.) In the case that Γ is connected, Postnikov [9, Theorem 12.2] showed that the $(p+q-2)$-dimensional volume of the edge polytope R_Γ times $(p+q-2)!$ equals $i(P_\Gamma^-)$ and $i(Q_\Gamma^-)$.

Now assume Γ is disconnected. Say $\Gamma = \Gamma_1 \cup \Gamma_2$ where Γ_1 and Γ_2 have vertex sets $\mathcal{I}_1 \cup \mathcal{J}_1$ and $\mathcal{I}_2 \cup \mathcal{J}_2$ for $\mathcal{I}_1 \cup \mathcal{I}_2 = \mathcal{I}$ and $\mathcal{J}_1 \cup \mathcal{J}_2 = \mathcal{J}$.

First observe that R_Γ is the convex hull of the union of the edge polytopes R_{Γ_1} and R_{Γ_2}. But these two polytopes have dimension at most $|\mathcal{I}_1| + |\mathcal{J}_1| - 2$ and $|\mathcal{I}_2| + |\mathcal{J}_2| - 2$, respectively, so R_Γ has dimension at most $|\mathcal{I}| + |\mathcal{J}| - 4 = p + q - 4$, and hence its $(p+q-2)$-dimensional volume is 0.

On the other hand, by the definition of P_Γ,

$$P_\Gamma \subset \Big\{x \in \mathbb{R}^q \, : \, \sum_{j \in \mathcal{J}_1} x_j = |\mathcal{I}_1|, \sum_{j \in \mathcal{J}_2} x_j = |\mathcal{I}_2|\Big\}.$$

so $\mathsf{dim}(P_\Gamma) < q-1 = \mathsf{dim}(\Delta_{[q]})$. Therefore $P_\Gamma^- = P_\Gamma - \Delta_{[q]} = \varnothing$ and $i(P_\Gamma^-) = 0$. The proof that $i(Q_\Gamma^-) = 0$ is analogous.

We have shown that (1.) = (2.) = (3.) = (4.). We have also shown that if Γ is disconnected this number is 0. On the other hand, if Γ is connected, then $\mathsf{dim} R_\Gamma = p+q-2$ by [7, Lemma 5.4], so its volume is nonzero.

(2. = 5. if Γ is connected.) The $(p+q-2)$-dimensional volume of R_Γ equals the degree of $V(I_\Gamma)$ by [15, Theorem 4.16]. $\square$

3.3 Our example, continued

Let us verify that $(2n-2)!\,\mathsf{MV}(G,G') = i(P^-_{\mathcal{I},\mathcal{J}}) = i(Q^-_{\mathcal{I},\mathcal{J}})$ for the graphs in Example 3.5. This case is small enough that we can do it by hand, and it illustrates the need for the machinery of Section 3.2. Here $n=6$, so $10!\,\mathsf{MV}(G,G')$ is the number of solutions to the system $\mathcal{E}(G,G')$

$$\mathcal{E}(G,G'): \quad \begin{cases} x_1 = \lambda_{12}\,x_2, & y_1 = \mu_{14}\,y_4, & \nu_{11}\,x_1y_1 + \cdots + \nu_{16}\,x_6y_6 = 0,\\ x_3 = \lambda_{34}\,x_4, & y_2 = \mu_{23}\,y_3, & \nu_{21}\,x_1y_1 + \cdots + \nu_{26}\,x_6y_6 = 0,\\ x_5 = \lambda_{56}\,x_6, & y_4 = \mu_{45}\,y_5, & \nu_{31}\,x_1y_1 + \cdots + \nu_{36}\,x_6y_6 = 0,\\ & y_5 = \mu_{56}\,y_6, & \\ x_1 = 1, & y_1 = 1. & \end{cases}$$

for a generic choice of coefficients. The first two columns of $\mathcal{E}(G,G')$ may be rewritten as

$$\begin{aligned} &1 = x_{12} := x_1 = \lambda_{12}\,x_2, \quad x_{34} := x_3 = \lambda_{34}\,x_4, \qquad x_{56} := x_5 = \lambda_{56}x_6\\ &1 = y_{1456} := y_1 = \mu_{14}\,y_4 = \mu_{14}\mu_{45}\,y_5, = \mu_{14}\mu_{45}\mu_{56}\,y_6, \quad y_{23} := y_2 = \mu_{23}\,y_3, \quad y_6 = y_6, \end{aligned}$$

so $\mathcal{E}(G,G')$ reduces to the following system of equations:

$$\mathcal{H}_{\mathcal{I},\mathcal{J}}: \quad \begin{cases} \eta_{11}\,x_{12}y_{1456} + \eta_{12}\,x_{12}y_{23} + \eta_{13}\,x_{34}y_{23} + \eta_{14}\,x_{34}y_{1456} + \eta_{15}\,x_{56}y_{1456} + \eta_{16}\,x_{56}y_{1456} = 0,\\ \eta_{21}\,x_{12}y_{1456} + \eta_{22}\,x_{12}y_{23} + \eta_{23}\,x_{34}y_{23} + \eta_{24}\,x_{34}y_{1456} + \eta_{25}\,x_{56}y_{1456} + \eta_{26}\,x_{56}y_{1456} = 0,\\ \eta_{31}\,x_{12}y_{1456} + \eta_{32}\,x_{12}y_{23} + \eta_{33}\,x_{34}y_{23} + \eta_{34}\,x_{34}y_{1456} + \eta_{35}\,x_{56}y_{1456} + \eta_{36}\,x_{56}y_{1456} = 0,\\ x_{12} = y_{145} = 1, \end{cases}$$

where each coefficient η_{ij} is obtained by multiplying ν_{ij} with the λs (or their inverses) along a path from i to $\min I(i)$ in G and the μs (or their inverses) along a path from j to $\min J(j)$ in G'. These coefficients are generic if the original λs, μs, and νs are sufficiently generic. If we write

$$z_1 = x_{12}y_{1456} = 1, \quad z_2 = x_{12}y_{23}, \quad z_3 = x_{34}y_{23}, \quad z_4 = x_{34}y_{1456}, \quad z_5 = x_{56}y_{1456}, \quad z_6 = x_{56}y_{1456}$$

we get a generic system of 3 equations in 5 unknowns $z_2, \ldots, z_6$. Solving this system, we obtain an expression for each of $z_2, \ldots, z_4$ as a linear function of z_5 and z_6.

Now, the z_is satisfy two equations

$$z_1z_3 = z_2z_4, \qquad z_5 = z_6$$

coming from the two even cycles formed by edges $1, 2, 3, 4$ and edges $5, 6$ in Γ, respectively. Thus z_2, z_3 and z_4 can be expressed linearly in terms of z_6, and the equation $z_1 z_3 = z_2 z_4$ turns into a quadratic equation satisfied by z_6, which has 2 solutions. Reversing the steps of our computation, we obtain 2 solutions to the original system. We conclude that $10!\,\mathsf{MV}(G, G') = 2$. This agrees with the fact that $P_\Gamma^- = \Delta_{ab} \subset \mathbb{R}^{abc}$ and $Q_\Gamma^- = \Delta_{AB} + \Delta_A \subset \mathbb{R}^{AB}$ each contain two lattice points.

The procedure above works for general acyclic graphs G and G' such that Γ is connected; the relations among the z_is are precisely given by the toric ideal I_Γ. The last step of the computation cannot be done by hand in general; instead, one needs to know the degree of I_Γ. We find it by computing the number of lattice points in $P_{\mathcal{I},\mathcal{J}}^-$ or in $Q_{\mathcal{I},\mathcal{J}}^-$ – whichever is easier.

3.4 The volume

We are finally ready to prove Theorem 1.1.

Theorem 1.1. The volume of the harmonic polytope is

$$\begin{aligned}\mathsf{Vol}(H_{n,n}) &= \sum_\Gamma \frac{i(P_\Gamma^-)}{(v(\Gamma)-2)!} \prod_{v \in V(\Gamma)} \mathsf{deg}(v)^{\mathsf{deg}(v)-2} \\ &= \sum_\Gamma \frac{\mathsf{deg}(X_\Gamma)}{(v(\Gamma)-2)!} \prod_{v \in V(\Gamma)} \mathsf{deg}(v)^{\mathsf{deg}(v)-2},\end{aligned}$$

summing over all connected bipartite multigraphs Γ on edge set $[n]$. Here $i(P_\Gamma^-)$ is the number of lattice points in the trimmed generalized permutahedron P_Γ^- of Γ, X_Γ is the projective embedding of the toric variety of Γ given by the toric ideal of Γ, $V(\Gamma)$ is the set of vertices of Γ, and $v(\Gamma) := |V(\Gamma)|$.

Proof. We use the notation of Sections 3.1 and 3.2. By (4) and Lemma 3.4 we have that

$$\begin{aligned}\mathsf{Vol}(H_{n,n}) &= \sum_{\substack{G,G' \\ \text{acyclic}}} \binom{2n-2}{G, G'; D_n} \mathsf{MV}(G, G') \\ &= \sum_{\substack{G,G' \\ \text{acyclic}}} \frac{(2n-2)!}{k!} \mathsf{MV}(G, G'),\end{aligned}$$

since the graphs G and G' have no repeated edges. Write Γ for the bipartite graph associated to G and G', abusing notation. Applying Proposition 3.10, and noting that $v(\Gamma) - 2 = p + q - 2 = k$, it follows that

$$\begin{aligned}\mathsf{Vol}(H_{n,n}) &= \sum_{\substack{G,G' \\ \text{acyclic}}} \mathsf{Vol}_{v(\Gamma)-2}(R_\Gamma) \\ &= \sum_{\substack{(G,G') \text{ acyclic} \\ \text{s.t. } \Gamma \text{ connected}}} \frac{i(P_\Gamma^-)}{(v(\Gamma)-2)!}.\end{aligned}$$

Since the summands on the right only depend on the partitions $\mathcal{I}, \mathcal{J}$ associated to G, G' we can combine the terms in as follows:

$$\mathsf{Vol}(H_{n,n}) = \sum_{\Gamma \text{ connected}} \frac{i(P_\Gamma^-)}{(v(\Gamma)-2)!} \cdot (\text{number of acyclic graphs } G, G' \text{ whose bipartite graph is } \Gamma).$$

Now, the edges of the bipartite graph Γ determine the labels $\mathcal{I} = \{I_1, \cdots, I_p\}$ and $\mathcal{J} = \{J_1, \cdots, J_q\}$ of the vertices of Γ; we need these to be the partitions of $[n]$ into the connected components of G and G', respectively.

We specify an acyclic graph G (resp. G') with components $\mathcal{I}$ (resp. $\mathcal{J}$) by specifying, for each $I \in \mathcal{I}$ (resp. $J \in \mathcal{J}$), a tree with $|I|$ (resp. $|J|$) vertices. There are $|I|^{|I|-2}$ (resp. $|J|^{|J|-2}$) such trees. By definition of $\Gamma_{\mathcal{I},\mathcal{J}}$, $\mathsf{deg}(I) = |I|$ for any $I \in \mathcal{I}$ (and similarly for any $J \in \mathcal{J}$). We thus conclude that

$$\mathsf{Vol}(H_{n,n}) \quad = \quad \sum_{\Gamma \text{ connected}} \frac{\mathsf{deg}(X_\Gamma)}{(|V(\Gamma)|-2)!} \prod_{v \in V(\Gamma)} \mathsf{deg}(v)^{\mathsf{deg}(v)-2}.$$

as desired. $\square$

Using Theorem 1.1 one can readily compute the volumes of the first few harmonic polytopes:

$$\mathsf{Vol}(H_{1,1}) = 1, \quad \mathsf{Vol}(H_{2,2}) = 3, \quad \mathsf{Vol}(H_{3,3}) = 33, \quad \mathsf{Vol}(H_{4,4}) = 2848/3.$$

3.5 Number of non-zero mixed volumes

In this section we compute the number of non-zero mixed volumes of the harmonic polytope, in its Minkowski sum decomposition (2). This is the number of summands that contribute to the volume of the harmonic polytope $H_{n,n}$ in (1). We do so with the help of the *Möbius algebra* of the partition lattice, which is denoted Π_n.[2]

If $\pi = \{B_1, \ldots, B_k\}$ is a set partition of $[n]$, we let $\ell(\pi) := k$ be the number of parts of π, and

$$t(\pi) := |B_1|^{|B_1|-2} \cdot \cdots \cdot |B_k|^{|B_k|-2}.$$

Let Π_n be the lattice of set partitions of $[n]$ ordered by refinement, so $\sigma \leq \tau$ if every block of τ is a union of blocks of σ.

Proposition 3.11. *The harmonic polytope $H_{n,n} = \mathsf{e}_{[n]} + \mathsf{f}_{[n]} + \sum_{i<j} \Delta_{ij} + \sum_{i<j} \Delta_{\overline{i}\overline{j}} + D_n$ has*

$$\begin{aligned} a_n \quad &:= \quad \textit{number of pairs of forests } (F_1, F_2) \textit{ on } [n] \textit{ such that } F_1 \cup F_2 \textit{ is connected} \\ &= \quad \sum_{\sigma \in \Pi_n} (-1)^{\ell(\sigma)} (\ell(\sigma)-1)! \Big(\sum_{\tau \leq \sigma} t(\tau) \Big)^2 \end{aligned}$$

non-zero mixed volumes.

Proof. A non-zero mixed volume cannot involve either of the summands $\mathsf{e}_{[n]}$ or $\mathsf{f}_{[n]}$, since the corresponding equations $\lambda\, x_1 \cdots x_n = 0$ and $\mu\, y_1 \cdots y_n = 0$ have no solutions on the torus for λ and μ generic. Thus we focus on the mixed volumes $\mathsf{MV}(G, G')$.

By Lemma 3.4 and Proposition 3.10, we have that $\mathsf{MV}(G, G') \neq 0$ if and only if G, G' are forests and the associated bipartite graph $\Gamma = \Gamma_{\mathcal{I},\mathcal{J}}$ is connected. Thus to prove the first statement we will show that $G \cup G'$ is connected if and only if Γ is connected.

Suppose that $G \cup G'$ is connected. A path

$$i_1 \to i_2 \to \ldots \to i_\ell$$

[2]This should not be confused with the permutohedron, which makes no further appearances in the paper.

in $G \cup G'$ gives rise to a path in Γ as follows. For $j = 1, \ldots, \ell$, replace the edge $i_j \to i_{j+1}$ in $G \cup G'$ with the edge $J(i_j) \to I(i_j) = I(i_{j+1})$ if $i_j i_{j+1} \in E(G)$, and with $I(i_j) \to J(i_j) = J(i_{j+1})$ in Γ if $i_j i_{j+1} \in E(G')$ Note that the resulting path can easily be modified into a path starting at $I(i_1)$ or $J(i_1)$ by adding or removing the edge $I(i_1)J(i_1)$; a similar modification works for $I(i_\ell)$ or $J(i_\ell)$. Now, to find a path between any two vertices of Γ, pick an element of each vertex, construct a path between these elements in $G \cup G'$, and use the procedure above to obtain a path between the desired vertices in Γ.

Conversely, suppose that Γ is connected and consider any two vertices i, i' of $G \cup G'$. Consider a path

$$P: \qquad I(i_1) \to J(i_1) = J(i_2) \to I(i_2) = I(i_3) \to \ldots \to J(i_{\ell-1}) = J(i_\ell) \qquad \text{in } \Gamma,$$

where $i_1 = i$ and $i_\ell = i'$. For each $1 \leq j \leq \ell - 1$, we have either $I(i_j) = I(i_{j+1})$ or $J(i_j) = J(i_{j+1})$; since these are connected components of G or G', we can find a path in either G or G' from i_j to i_{j+1}. We are then able to construct a path in $G \cup G'$ from i to i' by replacing each edge of the path P in Γ with a path from i_j to i_{j+1} in $G \cup G'$. This concludes the proof of the first equation.

Now, to choose a pair of forests (F_1, F_2) on $[n]$ such that $F_1 \cup F_2$ is connected, we first choose the set partitions $\pi_1 := \pi(F_1)$ and $\pi_2 = \pi(F_2)$, where $\pi(F)$ denotes the partition of $[n]$ given by the connected components of F. Notice that $F_1 \cup F_2$ is connected if and only if $\pi_1 \vee \pi_2 = \widehat{1}$ in the partition lattice. Having chosen the partitions π_1 and π_2, it simply remains to choose the forests F_1 and F_2 that give rise to them; there are $t(\pi_1)$ and $t(\pi_2)$ choices for those forests, respectively. It follows that

$$a_n = \sum_{\substack{\pi_1, \pi_2 \in \Pi_n \\ \pi_1 \vee \pi_2 = \widehat{1}}} t(\pi_1) t(\pi_2).$$

Now we compute in the *Möbius algebra* $A(\Pi_n)$ of Π_n; this is the real vector space with basis Π_n equipped with the bilinear multiplication given by the join of the lattice; in symbols,

$$A(\Pi_n) := \mathbb{R}\, \Pi_n \qquad \text{where} \qquad \sigma \cdot \tau := \sigma \vee \tau.$$

It follows from the definitions that

$$a_n = [\widehat{1}]\, T^2 \qquad \text{for} \qquad T := \sum_{\pi \in \Pi_n} t(\pi) \pi, \tag{7}$$

where $[\pi]\alpha$ denotes the coefficient of a set partition $\pi \in \Pi_n$ in an element $\alpha \in A(\Pi_n)$, when expressed in the standard basis.

As explained in [10, Section 3.9], it is useful to define the following elements of the Möbius algebra $A(\Pi_n)$:

$$\delta_\tau := \sum_{\sigma \geq \tau} \mu(\tau, \sigma) \sigma \qquad \text{for } \tau \in \Pi_n.$$

These elements form a basis for $A(\Pi_n)$ because Möbius inversion tells us that

$$\tau = \sum_{\sigma \geq \tau} \delta_\sigma, \qquad \text{for } \tau \in \Pi_n.$$

Furthermore, they are pairwise orthogonal idempotents:

$$\delta_\sigma \delta_\tau = \begin{cases} \delta_\sigma & \text{if } \sigma = \tau, \\ 0 & \text{otherwise,} \end{cases}$$

which makes them very useful for computations in $A(\Pi_n)$. We compute

$$\begin{aligned} T &= \sum_{\tau \in \Pi_n} \Big(t(\tau) \sum_{\sigma \geq \tau} \delta_\sigma \Big) \\ &= \sum_{\sigma \in \Pi_n} s(\sigma)\, \delta_\sigma, \end{aligned}$$

where

$$s(\sigma) := \sum_{\tau \leq \sigma} t(\tau) \qquad \text{for } \sigma \in \Pi_n.$$

Therefore, using the orthogonal idempotence of the δ_σs, we have

$$\begin{aligned} T^2 &= \sum_{\sigma \in \Pi_n} s(\sigma)^2 \delta_\sigma \\ &= \sum_{\sigma \in \Pi_n} \Big(s(\sigma)^2 \sum_{\tau \geq \sigma} \mu(\sigma, \tau) \tau \Big) \\ &= \sum_{\tau \in \Pi_n} \Big(\sum_{\sigma \leq \tau} \mu(\sigma, \tau) s(\sigma)^2 \Big) \tau \end{aligned}$$

It follows from (7) that

$$\begin{aligned} a_n &= \sum_{\sigma \in \Pi_n} \mu(\sigma, \widehat{1})\, s(\sigma)^2 \\ &= \sum_{\sigma \in \Pi_n} (-1)^{\ell(\sigma)} (\ell(\sigma) - 1)!\, s(\sigma)^2, \end{aligned}$$

using the facts that the interval $[\sigma, \widehat{1}]$ in the partition lattice Π_n is isomorphic to the smaller partition lattice $\Pi_{\ell(\sigma)}$ – because the coarsenings of σ are obtained by arbitrarily merging blocks of σ – and the Möbius number of the partition lattice Π_k is $\mu_{\Pi_k}(\widehat{0}, \widehat{1}) = (-1)^{k-1}(k-1)!$. $\square$

Using Proposition 3.11, one easily computes by hand the first values of the sequence:

$$a_1 = 1, \quad a_2 = 3, \quad a_3 = 39, \quad a_4 = 1242.$$

4 Acknowledgments

The first author would like to thank Graham Denham and June Huh for the very rewarding collaboration that led to the construction of the harmonic polytope.

References

[1] Federico Ardila, Graham Denham, and June Huh. Lagrangian geometry of matroids. Preprint, 2020. arXiv:2004.13116

[2] Federico Ardila and Caroline J. Klivans. The Bergman complex of a matroid and phylogenetic trees. *J. Combin. Theory Ser. B* **96** (2006) 38–49.

[3] Thomas Brylawski. The Tutte polynomial part I: General theory. In *Matroid theory and its applications*, volume 83 of *C.I.M.E. Summer Sch.*, pages 125–275. Springer, Heidelberg, 2010.

[4] D. N. Bernstein. The number of roots of a system of equations. *Funkcional. Anal. i Priložen.*, 9(3):1–4, 1975.

[5] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[6] Jeremy E. Dawson. A collection of sets related to the Tutte polynomial of a matroid. In *Graph theory, Singapore 1983*, volume 1073 of *Lecture Notes in Math.*, pages 193–204. Springer, Berlin, 1984.

[7] Jürgen Herzog, Takayuki Hibi, and Hidefumi Ohsugi. *Binomial ideals*, volume 279 of *Graduate Texts in Mathematics*. Springer, Cham, 2018.

[8] Lucía López de Medrano, Felipe Rincón, and Kristin Shaw. Chern-Schwartz-MacPherson cycles of matroids. *Proc. Lond. Math. Soc.* **120** (2020), 1–27.

[9] Alexander Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, (6):1026–1106, 2009.

[10] Richard P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.

[11] Robert MacPherson. Chern classes for singular algebraic varieties. *Ann. of Math.* **100** (1974), 423-432.

[12] Peter McMullen. Valuations and Euler-type relations on certain classes of convex polytopes. *Proc. London Math. Soc.* **3- 35** (1977), 113-135.

[13] C. Sabbah. Quelques remarques sur la géométrie des espaces conormaux. Number 130, pages 161–192. 1985. Differential systems and singularities (Luminy, 1983).

[14] Marie-Hélène Schwartz. Classes caractéristiques définies par une stratification d?une variété analytique complexe. I. *C. R. Acad. Sci. Paris* **260** (1965) 3262–3264.

[15] Bernd Sturmfels. *Gröbner bases and convex polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996.

[16] Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.