A generalization of Euler's $R \geq 2r$

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Let $\triangle ABC$ be a triangle with inradius r and circumradius R. A classical result due to Euler [1, p. 31], [2, p. 78] states that $R \geq 2r$. The purpose of this note is to extend this inequality, replacing the circumcircle of the triangle with an arbitrary ellipse that passes through its vertices. The main result of this note is the following.

Theorem 1 *Let $\triangle ABC$ be a triangle with inradius r, inscribed in an ellipse with constant k. (The ellipse is the locus of the points P such that the sum of the distances from P to two fixed points is k.) Then $k \geq 4r$.*

Notice that the circumcircle of $\triangle ABC$ is an ellipse whose foci both coincide with the circumcenter of the triangle, and whose constant is $2R$. Thus Euler's inequality is a special case of Theorem 1.

We start with the following lemma.

Lemma 2 *If a given circle of radius P contains a given triangle of inradius r, then $P \geq 2r$.*

Proof of Lemma 2.

Let S be the given circle of radius P and let O be its center. Let ABC be the given triangle with inradius r. It is clear that $3P \geq AO+BO+CO$. Now we prove that $AO + BO + CO \geq 6r$, which will finish the proof. Let D, E and F be the reflections of O across the sides BC, CA and AB respectively.

First assume that O is inside triangle ABC. Then the hexagon $AFBDCE$ has perimeter $2(AO + BO + CO)$ and area $2(ABC)$. From the isoperimetric theorem ([2, p. 63]) we know that, of all hexagons with a fixed perimeter p, the one with the largest area is the regular hexagon, which has area $p^2\sqrt{3}/24$. Therefore, $2(ABC) \leq (AO + BO + CO)^2\sqrt{3}/6$.

We also know that, of all triangles with fixed perimeter q, the one with the largest area is the equilateral triangle, which has area $q^2\sqrt{3}/36$. Therefore if s is the semiperimeter of triangle ABC, we have $sr = (ABC) \leq s^2\sqrt{3}/9$. Thus $s \geq 3\sqrt{3}\,r$ and $(ABC) = sr \geq 3\sqrt{3}\,r^2$.

Combining the two inequalities above, we get $AO+BO+CO \geq 6r$, which is the desired result.

The case where O is outside triangle ABC can be treated similarly. If O is inside angle BAC but outside triangle ABC, the hexagon $AFBOCE$ has perimeter $2(AO + BO + CO)$ and area greater than $2(ABC)$. We can then continue as above. Similarly, if O is inside the angle vertex opposite to angle BAC (so that A is inside triangle OBC) then the quadrilateral $OBDC$ has perimeter less than $2(AO + BO + CO)$ and area greater than $2(ABC)$. We can regard this quadrilateral as a degenerate hexagon, and again continue as above. $\square$

Proof of Theorem 1.

Let F_1 and F_2 be the foci of the ellipse. Let Γ_A be the circle with center A that passes through F_1, and define analogously Γ_B and Γ_C. Extend F_1A to meet Γ_A again at A_1, and define analogously B_1 and C_1. Let Γ be the circle with center F_2 and radius k. From the triangle inequality, we have that $F_2A_1 \leq F_2A+AA_1 = F_2A+AF_1 = k$, so A_1 is inside Γ. Similarly B_1 and C_1 are inside Γ. (In fact, Γ is internally tangent to Γ_A, Γ_B and Γ_C and contains them completely.) Also, notice that triangle $A_1B_1C_1$ is the homothety of triangle ABC with center F_1 and ratio 2, so its inradius is $2r$. Applying the lemma to triangle $A_1B_1C_1$ and circle Γ, we obtain that $k \geq 4r$. $\square$

References

[1] H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, New Mathematical Library, Mathematical Association of America, 1967.

[2] N. Kazarinoff, *Geometric Inequalities*, New Mathematical Library, Mathematical Association of America, 1961.