The equivariant volumes of the permutahedron

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Abstract

We consider the action of the symmetric group S_n on the permutahedron Π_n. We prove that if σ is a permutation of S_n which has m cycles of lengths $l_1, \ldots, l_m$, then the subpolytope of Π_n fixed by σ has normalized volume $n^{m-2}\gcd(l_1, \ldots, l_m)$.

1 Introduction

The n-permutahedron is the polytope in $\mathbb{R}^n$ whose vertices are the $n!$ permutations of $[n] := \{1, \ldots, n\}$:

$$\Pi_n := \operatorname{conv}\{(\pi(1), \pi(2), \ldots, \pi(n)) : \pi \in S_n\}.$$

The symmetric group S_n acts on $\Pi_n \subset \mathbb{R}^n$ by permuting coordinates; more precisely, a permutation $\sigma \in S_n$ acts on a point $x = (x_1, x_2, \ldots, x_n) \in \Pi_n$, by $\sigma \cdot x := (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(n)})$.

Definition 1.1. *The* subpolytope of the permutahedron Π_n fixed by a permutation σ of $[n]$ *is*

$$\Pi_n^\sigma = \{x \in \Pi_n \,:\, \sigma \cdot x = x\}.$$

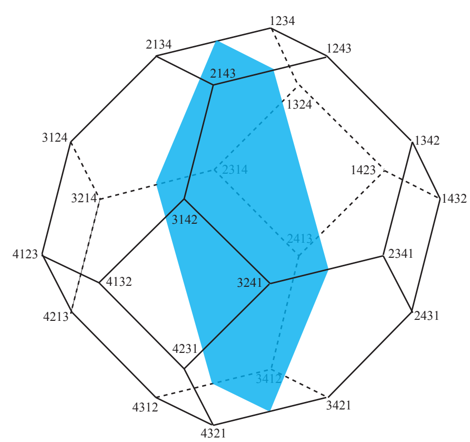

Figure 1: The subpolytope $\Pi_4^{(12)}$ of the permutahedron Π_4 fixed by $(12) \in S_4$ is a hexagon.

The main result of this short note is a generalization of the fact, due to Stanley [9], that $\operatorname{Vol}\Pi_n = n^{n-2}$.

Theorem 1.2. *If σ is a permutation of $[n]$ whose cycles have lengths $l_1, \ldots, l_m$, then the normalized volume of the subpolytope of Π_n fixed by σ is*

$$\operatorname{Vol}\Pi_n^\sigma = n^{m-2}\gcd(l_1, \ldots, l_m).$$

This is the first step towards describing the equivariant Ehrhart theory of the permutahedron, a question posed by Stapledon [11].

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The authors were supported by NSF Award DMS-1600609, NSF Award DMS-1440140 to MSRI, and the Simons Foundation (FA), an ARCS Foundation Fellowship (AS), and NSF Graduate Research Fellowship DGE-1247392 (ARVM).

1.1 Normalizing the volume

The permutahedron and its fixed subpolytopes are not full-dimensional; we must define their volumes carefully. We normalize volumes so that every primitive parallelotope has volume 1. This is the normalization under which the volume of Π_n equals n^{n-2}.

More precisely, let P be a d-dimensional polytope on an affine d-plane $L \subset \mathbb{Z}^n$. Assume L is integral, in the sense that $L \cap \mathbb{Z}^n$ is a lattice translate of a d-dimensional lattice Λ. We call a lattice d-parallelotope in L *primitive* if its edges generate the lattice Λ; all primitive parallelotopes have the same volume. Then we define the volume of a d-polytope P in L to be $\mathrm{Vol}(P) := \mathrm{EVol}(P)/\mathrm{EVol}(\Box)$ for any primitive parallelotope $\Box$ in L, where EVol denotes Euclidean volume.

The definition of $\mathrm{Vol}(P)$ makes sense even when P is not an integral polytope. This is important for us because the fixed subpolytopes of the permutahedron are not necessarily integral.

1.2 Notation

We identify each permutation $\pi \in S_n$ with the point $(\pi(1), \dots, \pi(n))$ in $\mathbb{R}^n$. When we write permutations in cycle notation, we do not use commas to separate the entries of each cycle. For example, we identify the permutation 246513 in S_6 with the point $(2, 4, 6, 5, 1, 3) \in \mathbb{R}^6$, and write it as $(1245)(36)$ in cycle notation.

Our main object of study is the fixed polytope Π_n^σ for a permutation $\sigma \in S_n$. We assume that σ has m cycles of lengths $l_1 \geq \cdots \geq l_m$. In fact, as we will show in Lemma 2.2, we may assume without losing generality that

$$\sigma = (1\ 2\ \dots\ l_1)(l_1+1\ \ l_1+2\ \dots\ l_1+l_2)\cdots(l_1+\cdots+l_{m-1}+1\ \dots\ n-1\ n).$$

We let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$, and $e_S := e_{s_1} + \cdots + e_{s_k}$ for $S = \{s_1, \dots, s_k\} \subseteq [n]$. Recall that the Minkowski sum of polytopes P, $Q \subset \mathbb{R}^n$ is the polytope $P + Q := \{p + q : p \in P, q \in Q\} \subset \mathbb{R}^n$. [5]

1.3 Organization

Section 2 is devoted to proving Theorem 2.12, which describes the fixed subpolytope Π_n^σ in terms of its vertices, its defining inequalities, and a Minkowski sum decomposition. Section 3 uses this to prove our main result, Theorem 1.2, that the normalized volume of Π_n^σ is $n^{m-2}\gcd(l_1, \dots, l_m)$. Section 4 contains some closing remarks.

2 Three descriptions of the fixed subpolytopes of the permutahedron

Proposition 2.1. *[14] The permutahedron Π_n can be described in the following three ways:*

1. *(Inequalities) It is the set of points $x \in \mathbb{R}^n$ satisfying*
   - *(a) $x_1 + x_2 + \cdots + x_n = 1 + 2 + \cdots + n$, and*
   - *(b) for any proper subset $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$,*

   $$x_{i_1} + x_{i_2} + \cdots + x_{i_k} \geq 1 + 2 + \cdots + k.$$

2. *(Vertices) It is the convex hull of the points $(\pi(1), \dots, \pi(n))$ as π ranges over the permutations of $[n]$.*
3. *(Minkowski sum) It is the Minkowski sum*

   $$\sum_{1 \leq j < k \leq n} [e_k, e_j] + \sum_{1 \leq k \leq n} e_k.$$

The n-permutahedron is $(n-1)$-dimensional and every permutation of $[n]$ is indeed a vertex.

Our first goal is to prove the analogous result for the fixed subpolytopes of Π_n; we do so in Theorem 2.12.

2.1 Standardizing the permutation

We define the *cycle type* of a permutation σ to be the partition of n consisting of the lengths $l_1 \geq \cdots \geq l_m$ of the cycles of σ.

Lemma 2.2. *The volume of Π_n^σ only depends on the cycle type of σ.*

Proof. Two permutations of S_n have the same cycle type if and only if they are conjugate [6]. For any two conjugate permutations σ and $\tau\sigma\tau^{-1}$ (where $\sigma, \tau \in S_n$) we have

$$\Pi_n^{\tau\sigma\tau^{-1}} = \tau \cdot \Pi_n^\sigma. \tag{1}$$

Every permutation $\tau \in S_n$ acts isometrically on $\mathbb{R}^n$ because S_n is generated by the transpositions $(i \;\; i+1)$ for $1 \leq i \leq n-1$, which act as reflections across the hyperplanes $x_i = x_{i+1}$. It follows from (1) that the fixed polytopes $\Pi_n^{\tau\sigma\tau^{-1}}$ and Π_n^σ have the same volume, as desired. $\square$

We wish to understand the various fixed subpolytopes of Π_n, and (1) shows that we can focus our attention on the subpolytopes Π_n^σ fixed by a permutation of the form

$$\sigma = (1 \;\, 2 \;\, \ldots \;\, l_1)(l_1+1 \;\, l_1+2 \;\, \ldots \;\, l_1+l_2)\cdots(l_1+\cdots+l_{m-1}+1 \;\, \ldots \;\, n-1 \;\, n) \tag{2}$$

for a partition $l_1 \geq l_2 \geq \cdots \geq l_m$ with $l_1 + \cdots + l_m = n$. We do so from now on.

2.2 The inequality description

Proposition 2.3. *For a permutation $\sigma \in S_n$, the fixed subpolytope Π_n^σ consists of the points $x \in \Pi_n$ satisfying $x_j = x_k$ for any j and k in the same cycle of σ.*

Proof. Suppose that $x \in \Pi_n^\sigma$. First, let h and i be adjacent entries in a cycle σ_a of σ, with $\sigma(h) = i$. Since $\sigma \cdot x = x$, we have

$$x_i = (\sigma \cdot x)_i = x_{\sigma^{-1}(i)} = x_h.$$

This holds for any adjacent entries of σ_a, so by transitivity $x_j = x_k$ for any two entries j, k of σ_a.

Conversely, suppose $x \in \Pi_n$ is such that $x_j = x_k$ whenever j and k are in the same cycle of σ. For any $1 \leq i \leq n$, let h be the index preceding i in the appropriate cycle of σ, so $\sigma(h) = i$. Then we have that $(\sigma \cdot x)_i = x_{\sigma^{-1}(i)} = x_h = x_i$. Since this holds for any index i, we have $\sigma \cdot x = x$ as desired. $\square$

Geometrically, Proposition 2.3 tells us that the fixed subpolytope Π_n^σ is the slice of Π_n cut out by the hyperplanes $x_j = x_k$ for all pairs j, k such that j and k are in the same cycle of σ. For example, the subpolytope of the permutahedron Π_4 fixed by the permutation (12) is the intersection of Π_4 with the hyperplane $x_1 = x_2$, as shown in Figure 1.

Corollary 2.4. *If a permutation σ of $[n]$ has m cycles then Π_n^σ has dimension $m-1$.*

Proof. Let $\sigma = \sigma_1 \cdots \sigma_m$ be the cycle decomposition of σ. A cycle $\sigma_j = (a_1 \;\, a_2 \;\, \cdots \;\, a_{l_j})$ of length l_j imposes $l_j - 1$ linear conditions on a point x in the fixed polytope, namely $x_{a_1} = x_{a_2} = \cdots = x_{a_{l_j}}$. Because σ has m cycles whose lengths add up to n, we have a total of $n - m$ such conditions, and they are linearly independent. The fixed subpolytope Π_n^σ is the transversal intersection of Π_n with these $n - m$ linearly independent hyperplanes, so $\dim \Pi_n^\sigma = \dim \Pi_n - (n-m) = m-1$. $\square$

2.3 Towards a vertex description

In this section we describe a set $\mathrm{Vert}(\sigma)$ of $m!$ points associated to a permutation σ of S_n. We will show in Theorem 2.12 that this is the set of vertices of the fixed polytope Π_n^σ. For a point $w \in \mathbb{R}^n$, let $\overline{w}$ be the average of the σ-orbit of w, that is,

$$\overline{w} := \frac{1}{|\sigma|} \sum_{i=1}^{|\sigma|} \sigma^i \cdot w, \tag{3}$$

where $|\sigma|$ is the order of σ as an element of the symmetric group S_n.

Definition 2.5. *Given $\sigma \in S_n$, we say a permutation $v = (v_1, \ldots, v_n)$ of $[n]$ is* σ-standard *if it satisfies the following property: for each cycle $(j_1\ j_2\ \cdots\ j_r)$ of σ, $(v_{j_1}, v_{j_2}, \ldots, v_{j_r})$ is a sequence of consecutive integers in increasing order. We define the set of* σ-vertices *to be*

$$\mathrm{Vert}(\sigma) := \{\overline{w} \,:\, w \text{ is a } \sigma\text{-standard permutation of } [n]\}.$$

These points should not be confused with the vertices of the ambient permutaheron Π_n. Let us illustrate this definition in an example and prove some preliminary results.

Example 2.6. For $\sigma = (1234)(567)(89)$, the σ-standard permutations in S_9 are

$$\begin{array}{ll} (1,2,3,4,\ 5,6,7,\ 8,9), & (1,2,3,4,\ 7,8,9,\ 5,6), \\ (4,5,6,7,\ 1,2,3,\ 8,9), & (3,4,5,6,\ 7,8,9,\ 1,2), \\ (6,7,8,9,\ 1,2,3,\ 4,5), & (6,7,8,9,\ 3,4,5,\ 1,2), \end{array}$$

and the corresponding σ-vertices are

$$\begin{array}{ll} \tfrac{1+2+3+4}{4}\, e_{1234} + \tfrac{5+6+7}{3}\, e_{567} + \tfrac{8+9}{2}\, e_{89}, & \tfrac{1+2+3+4}{4}\, e_{1234} + \tfrac{7+8+9}{3}\, e_{567} + \tfrac{5+6}{2}\, e_{89}, \\ \tfrac{4+5+6+7}{4}\, e_{1234} + \tfrac{1+2+3}{3}\, e_{567} + \tfrac{8+9}{2}\, e_{89}, & \tfrac{3+4+5+6}{4}\, e_{1234} + \tfrac{7+8+9}{3}\, e_{567} + \tfrac{1+2}{2}\, e_{89}, \\ \tfrac{6+7+8+9}{4}\, e_{1234} + \tfrac{1+2+3}{3}\, e_{567} + \tfrac{4+5}{2}\, e_{89}, & \tfrac{6+7+8+9}{4}\, e_{1234} + \tfrac{3+4+5}{3}\, e_{567} + \tfrac{1+2}{2}\, e_{89}. \end{array}$$

Let us give a more explicit description of $\overline{w}$ in general, and of the σ-vertices in particular, which will be important in the proof of Theorem 2.12.

Lemma 2.7. *For any $w \in \mathbb{R}^n$, the average of the σ-orbit of w is*

$$\overline{w} = \sum_{k=1}^{m} \frac{\sum_{j \in \sigma_k} w_j}{l_k}\, e_{\sigma_k}.$$

Proof. Let w_{σ_k} denote the projection of w to the coordinates in σ_k, so the ith coordinate of w_{σ_k} equals w_i if $i \in \sigma_k$ and 0 otherwise. Thus, $w = w_{\sigma_1} + \cdots + w_{\sigma_m}$ and

$$\overline{w} = \sum_{k=1}^{m} \overline{w_{\sigma_k}} = \sum_{k=1}^{m} \frac{1}{|\sigma|} \sum_{i=1}^{|\sigma|} \sigma^i w_{\sigma_k} = \frac{1}{|\sigma|} \sum_{k=1}^{m} \frac{|\sigma|}{|\sigma_k|} \sum_{i=1}^{|\sigma_k|} \sigma_k^i w_{\sigma_k},$$

because σ_k is the only cycle that acts on w_{σ_k} non-trivially, and $|\sigma|$ is a multiple of $|\sigma_k|$. For each cycle σ_k we have

$$\sum_{i=1}^{|\sigma_k|} \sigma_k^i w_{\sigma_k} = \Big(\sum_{j \in \sigma_k} w_j \Big) e_{\sigma_k},$$

from which the desired result follows. □

Notice that the entries of $\overline{w}$ within each cycle σ_k are constant, bearing witness to the fact that $\overline{w}$, being the average of a σ-orbit, must be in the fixed subpolytope Π_n^σ.

Corollary 2.8. *The set* $\mathrm{Vert}(\sigma)$ *of σ-vertices consists of the $m!$ points*

$$\overline{v_\prec} := \sum_{k=1}^{m} \Big(\frac{l_k + 1}{2} + \sum_{j \,:\, \sigma_j \prec \sigma_k} l_j \Big) e_{\sigma_k}$$

as $\prec$ ranges over the $m!$ possible linear orderings of $\sigma_1, \sigma_2, \ldots, \sigma_m$.

Proof. If $v = (v_1, \dots, v_n)$ is a σ-standard permutation, then for each cycle $(a_1\ a_2\ \dots\ a_r)$ of σ, $v_{a_1}, \dots, v_{a_r}$ is an increasing sequence of consecutive integers. The placement of these integers determines a linear ordering $\prec$ of $\sigma_1, \dots, \sigma_m$ as follows. For some cycle σ_a, the integers $1, 2, \dots, l_a$ are the coordinates in positions corresponding to σ_a. Set σ_a as the smallest cycle in $\prec$. Now, for some cycle σ_b, the integers $l_a + 1, l_a + 2, \dots, l_a + l_b$ are the coordinates in positions corresponding to σ_b. Set σ_b be the next smallest cycle in $\prec$. Continuing in this manner, we obtain a linear order on the set of cycles. Furthermore, any linear order $\prec$ of the cycles corresponds to a unique σ-standard permutation $v_\prec$ in this way.

Now, we can use Lemma 2.7 to compute $\overline{v_\prec}$: for each cycle σ_k, the set $\{v_i : i \in \sigma_k\}$ consists of the integers from $1 + \sum_{j\,:\,\sigma_j \prec \sigma_k} l_j$ to $l_k + \sum_{j\,:\,\sigma_j \prec \sigma_k} l_j$, whose average is $\frac{1}{2}(l_k + 1) + \sum_{j\,:\,\sigma_j \prec \sigma_k} l_j$. ☐

2.4 Towards a zonotope description

We will show in Theorem 2.12 that the fixed subpolytope Π_n^σ is the zonotope given by the following Minkowski sum.

Definition 2.9. *Let M_σ denote the Minkowski sum*

$$\begin{aligned} M_\sigma &:= \sum_{1 \le j < k \le m} [l_j e_{\sigma_k}, l_k e_{\sigma_j}] + \sum_{k=1}^{m} \frac{l_k + 1}{2} e_{\sigma_k} \\ &= \sum_{1 \le j < k \le m} [0, l_k e_{\sigma_j} - l_j e_{\sigma_k}] + \sum_{k=1}^{m} \left(\frac{l_k + 1}{2} + \sum_{j<k} l_j \right) e_{\sigma_k}. \end{aligned} \tag{4}$$

Recall that two polytopes P and Q are *combinatorially equivalent* if their posets of faces, partially ordered by inclusion, are isomorphic. They are *linearly equivalent* if there is a bijective linear function mapping P to Q. They are *normally equivalent* if they live in the same ambient vector space and have the same normal fan.

Lemma 2.10. *[5] If two polytopes are linearly or normally equivalent, then they are also combinatorially equivalent.*

Proposition 2.11. *The zonotope M_σ is combinatorially equivalent to the standard permutahedron Π_m, where m is the number of cycles of σ.*

Proof. Notice that $\{e_{\sigma_1}, \dots, e_{\sigma_m}\}$ is a basis for the subspace of $\mathbb{R}^n$ fixed by the action of σ:

$$(\mathbb{R}^n)^\sigma := \{x \in \mathbb{R}^n : x_j = x_k \text{ if } j \text{ and } k \text{ are in the same cycle of } \sigma\}.$$

Let $\{f_1, \dots, f_m\}$ be the standard basis for $\mathbb{R}^m$ and let $\phi : \mathbb{R}^m \to (\mathbb{R}^n)^\sigma$ be the map defined by $\phi(f_i) = \frac{1}{l_i} e_{\sigma_i}$ for $1 \le i \le m$. This map is bijective since it maps a basis of $\mathbb{R}^m$ to a basis of $(\mathbb{R}^n)^\sigma$. It follows that

$$\sum_{j<k} [f_k, f_j] \text{ is linearly equivalent to } \sum_{j<k} \left[\frac{1}{l_k} e_{\sigma_k}, \frac{1}{l_j} e_{\sigma_j} \right]. \tag{5}$$

Also, recall that the normal fan of a Minkowski sum $P_1 + \cdots P_s$ is the coarsest common refinement of the normal fans of $P_1, \dots, P_s$ [14, Prop. 7.12]. Therefore, scaling each summand in a Minkowski sum does not change the normal fan of the resulting polytope. It then follows that

$$\sum_{j<k} \left[\frac{1}{l_k} e_{\sigma_k}, \frac{1}{l_j} e_{\sigma_j} \right] \text{ is normally equivalent to } \sum_{j<k} l_j l_k \left[\frac{1}{l_k} e_{\sigma_k}, \frac{1}{l_j} e_{\sigma_j} \right] = \sum_{j<k} \left[l_j e_{\sigma_k}, l_k e_{\sigma_j} \right]. \tag{6}$$

Finally, since $\sum_{j<k} [f_k, f_j]$ and $\sum_{j<k} \left[l_j e_{\sigma_k}, l_k e_{\sigma_j} \right]$ are translates of Π_m and M_σ, respectively, the desired result follows from (5), (6), and Lemma 2.10. ☐

2.5 The three descriptions of the fixed subpolytope are equivalent

Theorem 2.12. *Let σ be a permutation of $[n]$ whose cycles $\sigma_1, \ldots, \sigma_m$ have respective lengths $l_1, \ldots, l_m$. The fixed subpolytope Π_n^σ can be described in the following four ways:*

0. It is the set of points x in the permutahedron Π_n such that $\sigma \cdot x = x$.

1. It is the set of points $x \in \mathbb{R}^n$ satisfying

(a) $x_1 + x_2 + \cdots + x_n = 1 + 2 + \cdots + n$,

(b) for any proper subset $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$,

$$x_{i_1} + x_{i_2} + \cdots + x_{i_k} \leq 1 + 2 + \cdots + k, \text{ and}$$

(c) for any i and j which are in the same cycle of σ, $x_i = x_j$.

2. It is the convex hull of the set $\mathrm{Vert}(\sigma)$ of σ-vertices, as described in Corollary 2.8.

3. It is the Minkowski sum M_σ of Definition 2.9, that is,

$$\sum_{1 \leq j < k \leq m} [l_j e_{\sigma_k}, l_k e_{\sigma_j}] + \sum_{k=1}^{m} \frac{l_k + 1}{2} e_{\sigma_k}.$$

Consequently, the fixed polytope Π_n^σ is a zonotope that is combinatorially isomorphic to the permutahedron Π_m. It is $(m-1)$-dimensional and every σ-vertex is indeed a vertex of Π_n^σ.

Proof. Description 0. is the definition of the fixed polytope Π_n^σ, and we already proved in Proposition 2.3 that description 1. is accurate. Recall that we denoted the polytopes described in 2. and 3 by $\mathrm{conv}(\mathrm{Vert}(\sigma))$ and M_σ, respectively. It remains to prove that

$$\Pi_n^\sigma = \mathrm{conv}(\mathrm{Vert}(\sigma)) = M_\sigma.$$

We proceed in three steps as follows:

$$A.\quad \mathrm{conv}(\mathrm{Vert}(\sigma)) \subseteq \Pi_n^\sigma \qquad B.\quad M_\sigma \subseteq \mathrm{conv}(\mathrm{Vert}(\sigma)) \qquad C.\quad \Pi_n^\sigma \subseteq M_\sigma$$

A. $\mathrm{conv}(\mathrm{Vert}(\sigma)) \subseteq \Pi_n^\sigma$: It suffices to show that Π_n^σ contains any point in $\mathrm{Vert}(\sigma)$, say

$$\overline{v_\prec} = \frac{1}{|\sigma|} \sum_{i=1}^{|\sigma|} \sigma^i \cdot v_\prec,$$

where $\prec$ is a total order of $\sigma_1, \ldots, \sigma_m$ and $v_\prec$ is the associated σ-standard permutation. Since $v_\prec$ is a vertex of Π_n, we conclude that $\sigma^i \cdot v_\prec$ is a vertex of Π_n for all i, and hence their average $\overline{v_\prec}$ is in Π_n. Also, since $\sigma^{|\sigma|} = 1$, we have that $\sigma \cdot \overline{v_\prec} = \overline{v_\prec}$. Therefore, $\overline{v_\prec}$ is in Π_n^σ by 0., as desired.

B. $M_\sigma \subseteq \mathrm{conv}(\mathrm{Vert}(\sigma))$: It suffices to show that any vertex of M_σ is in $\mathrm{Vert}(\sigma)$.

For a polytope $P \subset \mathbb{R}^n$ and a linear functional $c \in (\mathbb{R}^n)^*$, we let P_c denote the face of P where c is maximized. In particular, for any given vertex v of M_σ, consider a linear functional $c = (c_1, c_2, \ldots, c_n) \in (\mathbb{R}^n)^*$ such that $v = (M_\sigma)_c$ is the unique point in M_σ maximizing c. For $k = 1, \ldots, m$, let

$$c_{\sigma_k} := \frac{1}{l_k} \sum_{i \in \sigma_k} c_i.$$

We claim that
(a) $c_{\sigma_j} \neq c_{\sigma_k}$ for $j \neq k$, and
(b) $v = \overline{v_\prec}$ for the linear order $\prec$ on $\sigma_1, \sigma_2, \ldots, \sigma_m$ where $\sigma_j \prec \sigma_k$ if and only if $c_{\sigma_j} < c_{\sigma_k}$.
This will show that every vertex of M_σ is a σ-vertex, as desired.

(a) $c_{\sigma_j} \neq c_{\sigma_k}$ for $j \neq k$: Since Minkowski sums satisfy that $(P+Q)_c = P_c + Q_c$ [12, Equation 2.4], we have

$$v = (M_\sigma)_c = \sum_{j<k} [l_j e_{\sigma_k}, l_k e_{\sigma_j}]_c + \sum_{k=1}^{m} \frac{l_k+1}{2} e_{\sigma_k}. \tag{7}$$

This Minkowski sum is a point, so each summand $[l_j e_{\sigma_k}, l_k e_{\sigma_j}]_c$ must be a single point, equal to either $l_j e_{\sigma_k}$ or $l_k e_{\sigma_j}$. Therefore,

$$c(l_j e_{\sigma_k}) = l_j \sum_{i \in \sigma_k} c_i = l_j l_k c_{\sigma_k} \qquad \text{and} \qquad c(l_k e_{\sigma_j}) = l_k \sum_{i \in \sigma_j} c_i = l_j l_k c_{\sigma_j}$$

are distinct, hence $c_{\sigma_j} \neq c_{\sigma_k}$, as desired. We also see that

$$[l_j e_{\sigma_k}, l_k e_{\sigma_j}]_c = \begin{cases} l_j e_{\sigma_k} & \text{if } c_{\sigma_j} < c_{\sigma_k}, \\ l_k e_{\sigma_j} & \text{if } c_{\sigma_j} > c_{\sigma_k}. \end{cases} \tag{8}$$

(b) $v = \overline{v_\prec}$: The above argument shows that $c_{\sigma_1}, c_{\sigma_2}, \ldots, c_{\sigma_m}$ are strictly ordered; let $\prec$ be the corresponding linear order on $\sigma_1, \sigma_2, \ldots, \sigma_m$. Then (7) and (8) imply that

$$v = \sum_{k=1}^{m} \Big(\sum_{j \,:\, c_{\sigma_j} < c_{\sigma_k}} l_j \Big) e_{\sigma_k} + \sum_{k=1}^{m} \frac{l_k+1}{2} e_{\sigma_k} = \overline{v_\prec}$$

in light of Corollary 2.8; the desired result follows.

C. $\quad \Pi_n^\sigma \subseteq M_\sigma$: Any point $p \in \Pi_n^\sigma$ can be written as a convex combination $p = \sum_{\tau \in S_n} \lambda_\tau \tau$ of the $n!$ permutations of $[n]$, where $\lambda_\tau \geq 0$ for all τ and $\sum_{\tau \in S_n} \lambda_\tau = 1$. Recall from (3) that $\overline{w}$ represents the average of the σ-orbit of $w \in \mathbb{R}^n$. Since p is fixed by σ we have

$$p = \overline{p} = \sum_{\tau \in S_n} \lambda_\tau \overline{\tau}.$$

It follows that $\Pi_n^\sigma \subseteq \operatorname{conv}\{\overline{\tau} \,:\, \tau \in S_n\}$. Therefore, to show that $\Pi_n^\sigma \subseteq M_\sigma$, it suffices to show that $\overline{\tau} \in M_\sigma$ for all permutations τ. To do so, let us first derive an alternative expression for $\overline{\tau}$.

Let us begin with the vertex $\mathrm{id} = (1, 2, \ldots, n)$ of Π_n corresponding to the identity permutation. As described in Corollary 2.8, this is the σ-standard permutation corresponding to the order $\sigma_1 \prec \sigma_2 \prec \cdots \prec \sigma_m$, so

$$\overline{\mathrm{id}} = \sum_{k=1}^{m} \left(\frac{l_k+1}{2} + \sum_{j<k} l_j \right) e_{\sigma_k}. \tag{9}$$

Notice that this is the translation vector for the Minkowski sum of (4).

Now, let us compute $\overline{\tau}$ for any permutation τ. Let

$$l = \operatorname{inv}(\tau) = |\{(a,b) \,:\, 1 \leq a < b \leq n,\ \tau(a) > \tau(b)\}|$$

be the number of inversions of τ. Consider a minimal sequence $\mathrm{id} = \tau_0, \tau_1, \ldots, \tau_l = \tau$ of permutations such that τ_{i+1} is obtained from τ_i by exchanging the positions of numbers p and $p+1$, thus introducing a single new inversion without affecting any existing inversions. Such a sequence corresponds to a minimal factorization of τ as a product of simple transpositions $(p\ \ p+1)$ for $1 \leq p \leq n-1$. We have $\operatorname{inv}(\tau_i) = i$ for $1 \leq i \leq l$.

Now we compute $\overline{\tau}$ by analyzing how $\overline{\tau_i}$ changes as we introduce new inversions, using that

$$\overline{\tau} - \overline{\mathrm{id}} = (\overline{\tau_l} - \overline{\tau_{l-1}}) + \cdots + (\overline{\tau_1} - \overline{\tau_0}). \tag{10}$$

If $a < b$ are the positions of the numbers p and $p+1$ that we switch as we go from τ_i to τ_{i+1}, then regarding τ_i and τ_{i+1} as vectors in $\mathbb{R}^n$ we have

$$\tau_{i+1} - \tau_i = e_a - e_b.$$

If σ_j and σ_k are the cycles of σ containing a and b, respectively, we have

$$\overline{\tau_{i+1}} - \overline{\tau_i} = \overline{e_a} - \overline{e_b} = \frac{e_{\sigma_j}}{l_j} - \frac{e_{\sigma_k}}{l_k} = \frac{1}{l_j l_k}(l_k e_{\sigma_j} - l_j e_{\sigma_k}) \tag{11}$$

in light of Lemma 2.7. This is the local contribution to (10) that we obtain when we introduce a new inversion between a position a in cycle σ_j and a position b in cycle σ_k in our permutation. Notice that this contribution is 0 when $j = k$. Also notice that we will still have an inversion between positions a and b in all subsequent permutations, due to the minimality of the sequence. We conclude that

$$\overline{\tau} - \overline{\mathrm{id}} = \sum_{j<k} \frac{\mathrm{inv}_{j,k}(\tau)}{l_j l_k}(l_k e_{\sigma_j} - l_j e_{\sigma_k}) \tag{12}$$

where

$$\mathrm{inv}_{j,k}(\tau) = |\{(a,b) \,:\, 1 \leq a < b \leq n,\ a \in \sigma_j,\ b \in \sigma_k \text{ and } \tau(a) > \tau(b)\}|$$

is the number of inversions in τ between a position in σ_j and a position in σ_k for $j < k$.

Equations (9) and (12) give us an alternative description for $\overline{\tau}$. This description makes it apparent that $\overline{\tau} \in M_\sigma$: Notice that $|\sigma_j| = l_j$ and $|\sigma_k| = l_k$ imply that $0 \leq \mathrm{inv}_{j,k}(\tau) \leq l_j l_k$, so

$$\overline{\tau} - \overline{\mathrm{id}} \in \sum_{1 \leq j < k \leq n} [0, l_k e_{\sigma_j} - l_j e_{\sigma_k}];$$

combining this with (4) and (9) gives the desired result. □

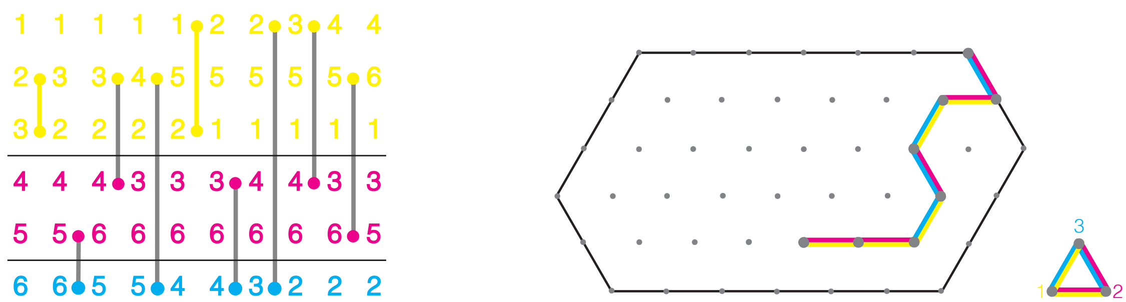

Figure 2: (a) A minimal sequence of permutations $\mathrm{id} = \tau_0, \tau_1, \ldots, \tau_9 = 461352$ adding one inversion at a time and (b) the corresponding path from $\overline{\mathrm{id}}$ to $\overline{\tau}$ in the zonotope M_σ.

Example 2.13. Figure 2 illustrates part C. of the proof above for $n = 6$, $\sigma = (123)(45)(6)$, and the permutation $\tau = 461352$. This permutation has $\mathrm{inv}(\tau) = 9$ inversions, and the columns of the left panel show a minimal sequence of permutations $\mathrm{id} = \tau_0, \tau_1, \ldots, \tau_9 = \tau$ where each τ_{i+1} is obtained from τ_i by swapping two consecutive numbers, thus introducing a single new inversion.

The rows of the diagram are split into three groups 1, 2, and 3, corresponding to the support of the cycles of σ. Out of the $\mathrm{inv}(\tau) = 9$ inversions of τ, there are $\mathrm{inv}_{1,2}(\tau) = 3$ involving groups 1 and 2, $\mathrm{inv}_{1,3}(\tau) = 2$ involve groups 1 and 3, and $\mathrm{inv}_{2,3}(\tau) = 2$ involving groups 2 and 3.

This sequence of permutations gives rise to a walk from $\overline{\mathrm{id}}$, which is the top right vertex of the zonotope M_σ, to $\overline{\tau}$. In the rightmost triangle, which is not drawn to scale, vertex i represents the point e_{σ_i}/l_i for $1 \leq i \leq 3$. Whenever two numbers in groups $j < k$ are swapped in the left panel, to get from permutation τ_i to τ_{i+1}, we take a step in direction $e_{\sigma_j}/l_j - e_{\sigma_k}/l_k$ in the right panel, to get from point $\overline{\tau_i}$ to $\overline{\tau_{i+1}}$. This is the direction of edge jk in the triangle, and its length is $1/l_j l_k$ of the length of the generator $l_k e_{\sigma_j} - l_j e_{\sigma_k}$ of the zonotope. Then

$$\overline{\tau} - \overline{\mathrm{id}} = \frac{3}{l_1 l_2}(l_2 e_{\sigma_1} - l_1 e_{\sigma_2}) + \frac{2}{l_1 l_3}(l_3 e_{\sigma_1} - l_1 e_{\sigma_3}) + \frac{2}{l_2 l_3}(l_3 e_{\sigma_2} - l_2 e_{\sigma_3}).$$

Since $3 = \mathrm{inv}_{1,2}(\tau) \leq l_1 l_2 = 6,\ 2 = \mathrm{inv}_{1,3}(\tau) \leq l_1 l_3 = 3$ and $2 = \mathrm{inv}_{2,3}(\tau) \leq l_2 l_3 = 2$, the resulting point $\overline{\tau}$ is in the zonotope M_σ.

3 The volumes of the fixed subpolytopes of Π_n

To compute the volume of the fixed subpolytope Π_n^σ we will use its description as a zonotope, recalling that a zonotope can be tiled by parallelotopes as follows. If A is a set of vectors, then $B \subseteq A$ is called a *basis* for A if B is linearly independent and $\mathrm{rank}(B) = \mathrm{rank}(A)$. We define the parallelotope $\Box B$ to be the Minkowski sum of the segments in B, that is,

$$\Box B := \Big\{ \sum_{b \in B} \lambda_b b \,:\, 0 \le \lambda_b \le 1 \text{ for each } b \in B \Big\}.$$

Theorem 3.1. *[4, 9, 14] Let $A \subset \mathbb{Z}^n$ be a set of lattice vectors of rank d.*

1. *The zonotope $Z(A)$ can be tiled using one translate of the parallelotope $\Box B$ for each basis B of A. Therefore, the volume of the d-dimensional zonotope $Z(A)$ is*

$$\mathrm{Vol}\,(Z(A)) = \sum_{\substack{B \subseteq A \\ B \text{ basis}}} \mathrm{Vol}\,(\Box B).$$

2. *For each $B \subset \mathbb{Z}^n$ of rank d, $\mathrm{Vol}(\Box B)$ equals the index of $\mathbb{Z}B$ as a sublattice of $(\mathrm{span}\, B) \cap \mathbb{Z}^n$. Using the vectors in B as the columns of an $n \times d$ matrix, $\mathrm{Vol}(B)$ is the greatest common divisor of the minors of rank d.*

By Theorem 2.12, the fixed polytope Π_n^σ is a translate of the zonotope generated by the set

$$F_\sigma = \left\{ l_k e_{\sigma_j} - l_j e_{\sigma_k}\,;\, 1 \le j < k \le m \right\}.$$

This set of vectors has a nice combinatorial structure, which will allow us to describe the bases B and the volumes $\mathrm{Vol}\,(\Box B)$ combinatorially. We do this in the next two lemmas. For a tree T whose vertex set is $[m]$, let

$$\begin{aligned} F_T &= \left\{ l_k e_{\sigma_j} - l_j e_{\sigma_k} \,:\, j < k \text{ and } jk \text{ is an edge of } T \right\}, \\ E_T &= \left\{ \frac{e_{\sigma_j}}{l_j} - \frac{e_{\sigma_k}}{l_k} \,:\, j < k \text{ and } jk \text{ is an edge of } T \right\}. \end{aligned}$$

Lemma 3.2. *The vector configuration*

$$F_\sigma := \left\{ l_k e_{\sigma_j} - l_j e_{\sigma_k} \,:\, 1 \le j < k \le m \right\}$$

has exactly m^{m-2} bases: they are the sets F_T as T ranges over the trees on $[m]$.

Proof. The vectors in F_σ are positive scalar multiples of the vectors in

$$E_\sigma = \left\{ \frac{e_{\sigma_j}}{l_j} - \frac{e_{\sigma_k}}{l_k} \,:\, 1 \le j < k \le m \right\},$$

which are the images of the vector configuration

$$A_{m-1}^+ = \{ f_j - f_k \,:\, 1 \le j < k \le m \}$$

under the bijective linear map

$$\begin{aligned} \phi : \mathbb{R}^m &\to (\mathbb{R}^n)^\sigma \\ f_i &\mapsto \frac{e_{\sigma_i}}{l_i} \end{aligned} \tag{13}$$

studied in Proposition 2.11. The set of vectors A_{m-1}^+ is well-studied. It consists of the positive roots of the Lie algebra $\mathfrak{gl}_n$; its bases are known [2] to correspond to the trees T on $[m]$, and there are m^{m-2} of them by Cayley's formula [3]. It follows that the bases of F_σ are precisely the sets F_T as T ranges over those m^{m-2} trees. $\Box$

Lemma 3.3. *For any tree T on $[m]$ we have*

$$\begin{aligned} 1. \quad \mathrm{Vol}(\Box F_T) &= \prod_{i=1}^{m} l_i^{\deg_T(i)} \mathrm{Vol}(E_T), \\ 2. \quad \mathrm{Vol}(\Box E_T) &= \frac{\gcd(l_1, \ldots, l_m)}{l_1 \cdots l_m}, \end{aligned}$$

where $\deg_T(i)$ *is the number of edges containing vertex i in T.*

Proof. 1. Since $l_k e_{\sigma_j} - l_j e_{\sigma_k} = l_j l_k \left(\frac{e_{\sigma_j}}{l_j} - \frac{e_{\sigma_k}}{l_k}\right)$ for each edge jk of T, and volumes scale linearly with respect to each edge length of a parallelotope, we have

$$\begin{aligned} \mathrm{Vol}(\Box F_T) &= \Big(\prod_{jk \text{ edge of T}} l_j l_k \Big) \mathrm{Vol}(\Box E_T) \\ &= \prod_{i=1}^{m} l_i^{\deg_T(i)} \mathrm{Vol}(\Box E_T) \end{aligned}$$

as desired.

2. The parallelotopes $\Box E_T$ are the images of the parallelotopes $\Box A_T$ under the linear bijective map ϕ of (13), where

$$A_T := \{f_j - f_k \, : \, j < k, \, jk \text{ is an edge of } T\}.$$

Since the vector configuration $\{f_j - f_k \, : \, 1 \le j < k \le m\}$ is unimodular [8], all parallelotopes $\Box A_T$ have unit volume. Therefore, the parallelotopes $\Box E_T = \phi(\Box A_T)$ have the same normalized volume, so $\mathrm{Vol}(E_T)$ is independent of T.

It follows that we can use any tree T to compute $\mathrm{Vol}(E_T)$ or, equivalently, $\mathrm{Vol}(F_T)$. We choose the tree $T = \mathrm{Claw}_m$ with edges $1m, 2m, \ldots, (m-1)m$. Writing the $m-1$ vectors of

$$F_{\mathrm{Claw}_m} = \{l_m e_{\sigma_i} - l_i e_{\sigma_m} \, : \, 1 \le i \le m-1\}$$

as the columns of an $n \times (m-1)$ matrix, then $\mathrm{Vol}(F_{\mathrm{Claw}_m})$ is the greatest common divisor of the non-zero maximal minors of this matrix. This quantity does not change when we remove duplicate rows; the result is the $m \times (m-1)$ matrix

$$\begin{bmatrix} l_m & 0 & 0 & \cdots & 0 \\ 0 & l_m & 0 & \cdots & 0 \\ 0 & 0 & l_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & l_m \\ -l_1 & -l_2 & -l_3 & \cdots & -l_{m-1} \end{bmatrix}.$$

This matrix has m maximal minors, whose absolute values equal $l_m^{m-2} l_1, l_m^{m-2} l_2, \ldots l_m^{m-2} l_{m-1}, l_m^{m-1}$. Therefore,

$$\mathrm{Vol}(\Box F_{\mathrm{Claw}_m}) = l_m^{m-2} \gcd(l_1, \ldots, l_{m-1}, l_m)$$

and part 1 then implies that

$$\mathrm{Vol}(\Box E_{\mathrm{Claw}_m}) = \frac{\mathrm{Vol}(\Box F_{\mathrm{Claw}_m})}{l_1 \cdots l_{m-1} l_m^{m-1}} = \frac{\gcd(l_1, \ldots, l_m)}{l_1 \cdots l_m}$$

as desired. $\Box$

Lemma 3.4. *For any positive integer $m \geq 2$ and unknowns $x_1, \ldots, x_m$, we have*

$$\sum_{T \text{ tree on } [m]} \prod_{i=1}^{m} x_i^{\deg_T(i)-1} = (x_1 + \cdots + x_m)^{m-2}.$$

Proof. We derive this from the analogous result for rooted trees [10, Theorem 5.3.4], which states that

$$\sum_{\substack{(T,r) \text{ rooted} \\ \text{tree on } [m]}} \prod_{i=1}^{m} x_i^{\mathrm{children}_{(T,r)}(i)} = (x_1 + \cdots + x_m)^{m-1}$$

where $\mathrm{children}_{(T,r)}(v)$ counts the children of v; that is, the neighbors of v which are not on the unique path from v to the root r.

Notice that

$$\mathrm{children}_{(T,r)}(i) = \begin{cases} \deg_T(i) - 1 & \text{if } i \neq r, \\ \deg_T(i) & \text{if } i = r. \end{cases}$$

Therefore,

$$\begin{aligned} \sum_{\substack{(T,r) \text{ rooted} \\ \text{tree on } [m]}} \prod_{i=1}^{m} x_i^{\mathrm{children}_{(T,r)}(i)} &= \sum_{r=1}^{m} \left(\sum_{\substack{(T,r) \text{ tree on } [m] \\ \text{rooted at } r}} x_r \prod_{i=1}^{m} x_i^{\deg_T(i)-1} \right) \\ &= \left(\sum_{T \text{ tree on } [m]} \prod_{i=1}^{m} x_i^{\deg_T(i)-1} \right) (x_1 + \cdots + x_m) \end{aligned}$$

from which the desired result follows. □

Theorem 1.2 . *If σ is a permutation of $[n]$ whose cycles have lengths $l_1, \ldots, l_m$, then the normalized volume of the subpolytope of Π_n fixed by σ is*

$$\mathrm{Vol}\, \Pi_n^\sigma = n^{m-2} \gcd(l_1, \ldots, l_m).$$

Proof. Since Π_n^σ is a translate of the zonotope for the lattice vector configuration

$$F_\sigma := \left\{ l_k e_{\sigma_j} - l_j e_{\sigma_k} \,:\, 1 \leq j < k \leq m \right\},$$

we invoke Theorem 3.1. Using Lemmas 3.2, 3.3, and 3.4, it follows that

$$\begin{aligned} \mathrm{Vol}\, \Pi_n^\sigma &= \sum_{T \text{ tree on } [m]} \mathrm{Vol}(\square F_T) \\ &= \sum_{T \text{ tree on } [m]} \prod_{i=1}^{m} l_i^{\deg_T(i)-1} \gcd(l_1, \ldots, l_m) \\ &= (l_1 + \cdots + l_m)^{m-2} \gcd(l_1, \ldots, l_m), \end{aligned}$$

as desired. □

When $\sigma = \mathrm{id}$ is the identity permutation, the fixed polytope is $\Pi_n^{\mathrm{id}} = \Pi_n$, and we recover Stanley's result that $\mathrm{Vol}\, \Pi_n = n^{n-2}$. [9]

4 Closing remarks

4.1 Subpolytopes of Π_n fixed by a subgroup of S_n

One might ask, more generally, for the subpolytope of Π_n fixed by a subgroup of H in S_n; that is,

$$\Pi_n^H = \{ x \in \Pi_n \,:\, \sigma \cdot x = x \text{ for all } \sigma \in H \}.$$

It turns out that this more general definition leads to the same family of subpolytopes of Π_n.

Lemma 4.1. *For every subgroup H of S_n there is a permutation σ of S_n such that $\Pi_n^H = \Pi_n^\sigma$.*

Proof. Let $\{\sigma_1, \ldots, \sigma_r\}$ be a set of generators for H. Notice that a point $p \in \mathbb{R}^n$ is fixed by H if and only if it is fixed by each one of these generators. For each generator σ_t, the cycles of σ_t form a set partition π_t of $[n]$. Furthermore, a point $x \in \mathbb{R}^n$ is fixed by σ_t if and only if $x_j = x_k$ whenever j and k are in the same part of π_t.

Let $\pi = \pi_1 \vee \cdots \vee \pi_r$ in the lattice of partitions of $[n]$; the partition π is the finest common coarsening of $\pi_1, \ldots, \pi_r$. Then $x \in \mathbb{R}^n$ is fixed by each one of the generators of H if and only if $x_j = x_k$ whenever j and k are in the same part of π. Therefore, we may choose any permutation σ of $[n]$ whose cycles are supported on the parts of π, and we will have $\Pi_n^H = \Pi_n^\sigma$, as desired. $\square$

Example 4.2. Consider the subpolytope of Π_9 fixed by the subgroup $H = \langle (173)(46)(89), (27)(68) \rangle$ of S_9. To be fixed by the two generators of H, a point $x \in \mathbb{R}^9$ must satisfy

$$\begin{aligned} \sigma_1 = (173)(46)(89): &\qquad x_1 = x_7 = x_3, \quad x_4 = x_6, \quad x_8 = x_9, \\ \sigma_2 = (27)(68): &\qquad x_2 = x_7, \quad x_6 = x_8, \end{aligned}$$

corresponding to the partitions $\pi_1 = 137|2|46|5|89$ and $\pi_2 = 1|27|3|4|5|68|9$. Combining these conditions gives

$$x_1 = x_2 = x_3 = x_7, \quad x_4 = x_6 = x_8 = x_9,$$

which corresponds to the join $\pi_1 \vee \pi_2 = 1237|4689|5$. For any permutation σ whose cycles are supported on the parts of $\pi_1 \vee \pi_2$, such as $\sigma = (1237)(4689)$, we have $\Pi_9^H = \Pi_9^\sigma$.

4.2 Lattice point enumeration and equivariant Ehrhart theory

Theorem 1.2 is the first step towards describing the equivariant Ehrhart theory of the permutahedron, a question posed by Stapledon [11]. To carry out this larger project, we need to compute the Ehrhart quasipolynomial of Π_n^σ, which counts the lattice points in its integer dilates:

$$L_{\Pi_n^\sigma}(t) := |\, t\,\Pi_n^\sigma \cap \mathbb{Z}^n| \qquad \text{for } t \in \mathbb{N}.$$

New difficulties arise in this question; let us briefly illustrate some of them.

When all cycles of σ have odd length, Theorem 2.12.3 shows that Π_n^σ is a lattice zonotope. In this case, it is not much more difficult to give a combinatorial formula for the Ehrhart polynomial, using the fact that $L_{\Pi_n^\sigma}(t)$ is an evaluation of the arithmetic Tutte polynomial of the corresponding vector configuration [1, 4].

In general, Π_n^σ is a half-integral zonotope. Therefore, the even part of its Ehrhart quasipolynomial is also an evaluation of an arithmetic Tutte polynomial, and can be computed as above. However, the odd part of its Ehrhart quasipolynomial is more subtle. If we translate Π_n^σ to become integral, we can lose and gain lattice points in the interior and on the boundary, in ways that depend on number-theoretic properties of the cycle lengths.

Some of these subtleties already arise in the simple case when Π_n^σ is a segment; that is, when σ has only two cycles of lengths l_1 and l_2. For even t, we simply have

$$L_{\Pi_n^\sigma}(t) = \gcd(l_1, l_2)t + 1.$$

However, for odd t we have

$$L_{\Pi_n^\sigma}(t) = \begin{cases} \gcd(l_1, l_2)t + 1 & \text{if } l_1 \text{ and } l_2 \text{ are both odd,} \\ \gcd(l_1, l_2)t & \text{if } l_1 \text{ and } l_2 \text{ have different parity,} \\ \gcd(l_1, l_2)t & \text{if } l_1 \text{ and } l_2 \text{ are both even and they have the same 2-valuation,} \\ 0 & \text{if } l_1 \text{ and } l_2 \text{ are both even and they have different 2-valuations,} \end{cases}$$

where the 2-*valuation* of a positive integer is the highest power of 2 dividing it.

In higher dimensions, additional obstacles arise. Describing the equivariant Ehrhart theory of the permutahedron is the subject of an upcoming project.

5 Acknowledgments

Some of the results of this paper are part of the Master's theses of AS (under the supervision of FA) and ARVM (under the supervision of FA and Matthias Beck) at San Francisco State University [7, 13]. We are grateful to Anastasia Chavez, John Guo, Andrés Rodríguez, and Nicole Yamzon for their valuable feedback during our group research meetings, and the mathematics department at SFSU for providing a wonderful environment to produce this work. Part of this project was carried out while FA was a Simons Research Professor at the Mathematical Sciences Research Institute; he thanks the Simons Foundation and MSRI for their support. ARVM thanks Matthias Beck and Benjamin Braun for the support and fruitful conversations.

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