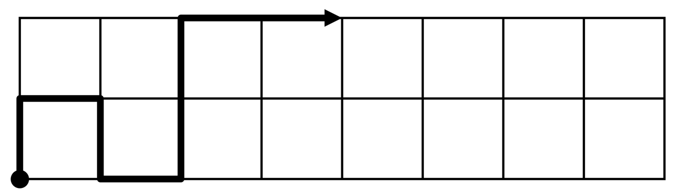
The configuration space of a robotic arm in a tunnel

Federico Ardila[1] Hanner Bastidas[2] Cesar Ceballos[3] John Guo[1]

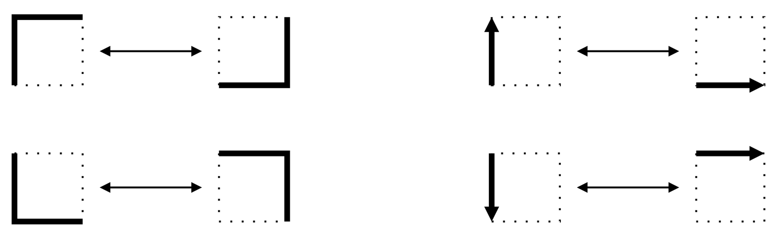
1. San Francisco State University 2. Universidad del Valle 3. Universität Wien

The Robotic Arm

We consider a robotic arm $R_{2,n}$ of length n moving in a rectangular tunnel of width 2 without self-intersecting. The robot consists of n links of unit length, attached sequentially, and its base is affixed to the lower left corner:



The robot starts in a fully horizontal position, and is free to move using two kinds of local moves:
- *Switching corners:* Two consecutive links facing different directions swap directions.
- *Flipping the end:* The last link of the robot rotates $90°$.



Problem 1. Move the robot **optimally** from one position to another.

CAT(0) Cubical Complexes

Definition. The *configuration space* $\mathcal{S}_n$ of the robot $R_{2,n}$ is the following cubical complex.
- vertices: states of the robot.
- edges: moves between two states.
- k-cubes: k-tuples of moves which can be performed simultaneously.

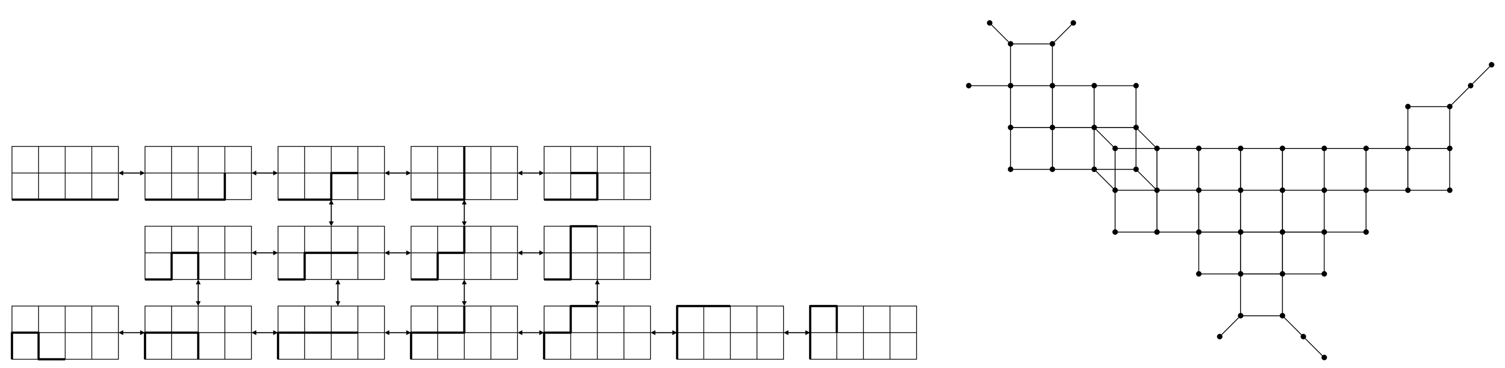
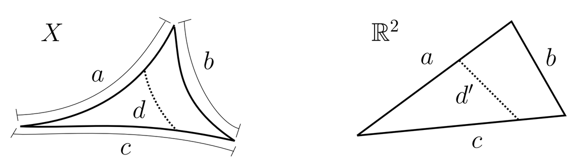


Figure 1: The configuration spaces $\mathcal{S}_4$ (with states shown) and $\mathcal{S}_6$.

Definition. A metric space X is said to be *CAT(0)* if:
- there is a unique geodesic (shortest) path between any two points in X, and
- X has non-positive global curvature; *i.e.* all its triangles are thin:



Theorem 1. [2] If the configuration space of a robot is CAT(0), then Problem 1 can be solved for that robot.

Checking whether an arbitrary space is CAT(0) is not easy; but for cubical complexes it can be done completely combinatorially and topologically:

Theorem. (Gromov, [6]) A cubical complex is $\mathrm{CAT}(0)$ if and only if it is simply connected and the link of every vertex is a flag simplicial complex.

Things are even easier if X is a "robotic" cubical complex, because:
- the link of every vertex is always flag [1] and
- X is simply connected if and only if it is contractible. [4]

Face Enumeration and the Euler Characteristic

By the previous observations, in order to be able to solve Problem 1, it suffices to show that $\mathcal{S}_n$ is contractible. We begin with some preliminary evidence.

Theorem 1. The Euler characteristic of the configuration space $\mathcal{S}_n$ equals 1.

Sketch of proof. A d-cube in the configuration space $\mathcal{S}_n$ has 2^d vertices. If one superimposes the corresponding 2^d positions of the robotic arm, one obtains a sequence of edges and squares, possibly including a "claw" in the last position. The number of squares is d, corresponding to the d physically independent moves. The weight of this *partial state* is $x^n y^d$.

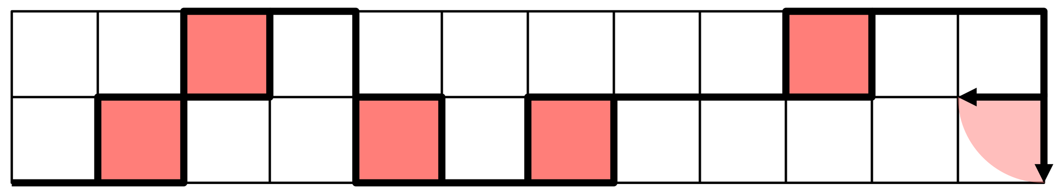


Figure 2: A partial state corresponding to a 6-cube in the configuration space $\mathcal{S}_{20}$.

We compute the generating function for partial states according to their weight, by noticing that we can uniquely "factor" any partial state into a concatenation of irreducible factors.

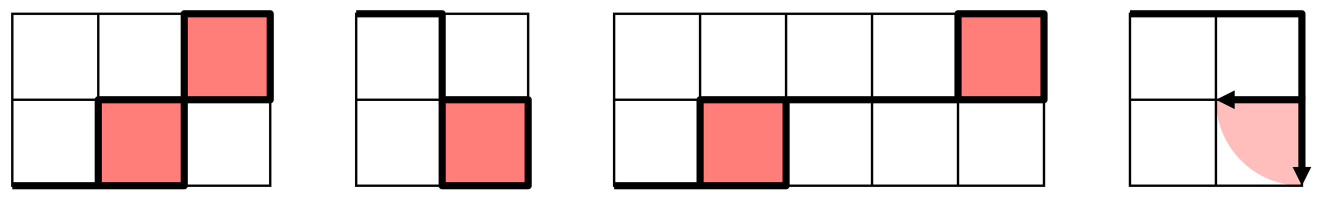


Figure 3: The partial state of Figure 2 factored into irreducibles.

We enumerate the irreducibles, and analyze how they can be glued together. We obtain:

Theorem 2. Let $\mathcal{S}_n$ be the configuration space for the robot of length n moving in a rectangular tunnel of width 2. If $c_{n,d}$ denotes the number of d-dimensional cubes in $\mathcal{S}_n$ then $C(x,y) = \sum_{n,d\geq 0} c_{n,d}\, x^n y^d$ equals

$$\frac{1+x^2+2x^3-x^4+xy+x^2y+4x^3y+x^4y+x^3y^2+2x^4y^2+x^5y^2}{1-2x+x^2-x^3-x^4-2x^4y-2x^5y-x^5y^2-x^6y^2}.$$

Proof that Theorem 2 ⇒ Theorem 1: The Euler characteristic of $\mathcal{S}_n$ is $\chi(\mathcal{S}_n) = \sum_{d\geq 0}(-1)^d c_{n,d}$, so the generating function for $\chi(\mathcal{S}_n)$ is given by substituting $y=-1$ into our generating function above. We obtain an expected but still beautiful miracle of cancellation:

$$\sum_{n\geq 0}\chi(\mathcal{S}_n)x^n = C(x,-1) = \frac{1-x-x^3+x^5}{1-2x+x^2-x^3+x^4+x^5-x^6} = \frac{1}{1-x} = 1+x+x^2+x^3+\ldots.$$

References

[1] A. Abrams and R. Ghrist. State complexes for metamorphic robots. *Int. J. Robotics Res.* **23** (2004) 811-826.

[2] F. Ardila, T. Baker, and R. Yatchak. Moving robots efficiently using the combinatorics of $\mathrm{CAT}(0)$ cubical complexes. *Adv. in Appl. Math.* **48** (2012) 142-163.

[3] F. Ardila, M. Owen, and S. Sullivant. Geodesics in $\mathrm{CAT}(0)$ cubical complexes. *SIAM J. Discrete Math.* **28-2** (2014) 986-1007.

[4] M. Bridson and A. Haefligher. Metric spaces of non-positive curvature. Springer-Verlag, Berlin, Heidelberg, 1999.

[5] R. Ghrist and V. Peterson. The geometry and topology of reconfiguration. *Adv. in Appl. Math.* **38** (2007) 302-323.

[6] M. Gromov. Hyperbolic groups. In Essays in group theory, volume 8 of *Math. Sci. Res. Inst. Publ.* 75263. Springer, New York, 1987.

CAT(0) Cubical Complexes and PIPs

Having obtained preliminary evidence that $\mathcal{S}_n$ is CAT(0), we proceed to prove it. Instead of using Gromov's topological combinatorial characterization of CAT(0) cubical complexes, we use the following purely combinatorial alternative.

Definition. A *poset with inconsistent pairs (PIP)* is a poset P together with a collection of *inconsistent pairs*, denoted $p \cdots q$ (where $p \neq q$), such that $p \cdots q$ and $q < q'$ implies $p \cdots q'$.

Theorem 3. [3] There is a bijection between posets with inconsistent pairs (PIPs) and rooted $\mathrm{CAT}(0)$ cube complexes.

Thus to prove a cubical complex X is CAT(0) it "simply" suffices to identify its PIP! The PIP is much simpler and serves as a "remote control" to navigate the space X and solve Problem 1.

The Remote Control: The Coral PIP

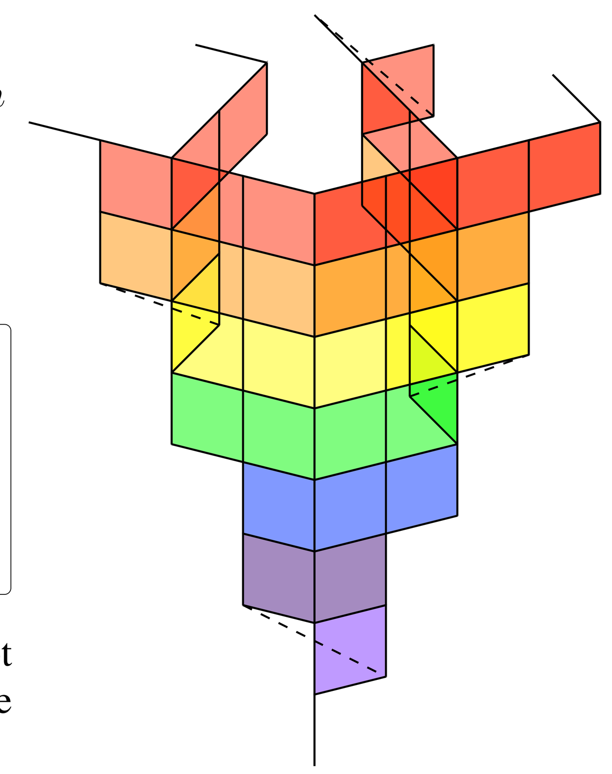
More generally, we study the robotic arm $R_{m,n}$ of length n in a tunnel **of any width** m.

Definition. A *coral snake* λ of height at most m is a path of unit squares, colored alternatingly black and red (starting with black), inside the tunnel of width m such that:
1. The snake λ starts at the bottom left of the tunnel, and takes steps up, down, and right.
2. Let λ turn from a vertical V_1 to a horizontal H to a vertical V_2 at corners C_1 and C_2. Then V_1 and V_2 face the same direction if and only if C_1 and C_2 have the same color.

The *length* $l(\lambda)$ is the number of unit squares of λ, and the *height* $h(\lambda)$ is the number of rows it touches, and the *width* $w(\lambda)$ is the number of columns it touches. We write $\lambda \subseteq \mu$ if λ is an initial sub-snake of μ, obtained by restricting to the first k cells of μ for some k.

Define the *coral PIP* $C_{m,n}$ as follows:

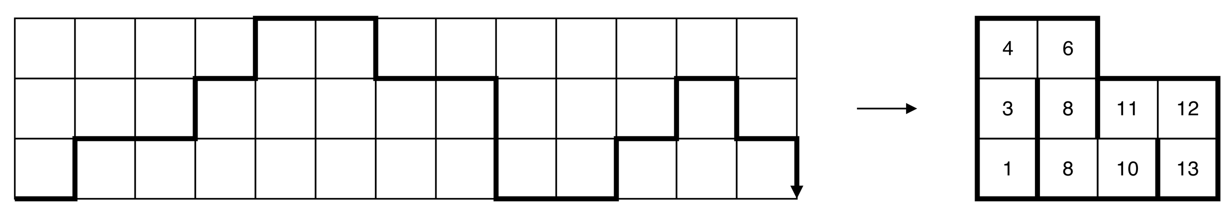
1. Elements: pairs (λ, s) of a coral snake λ with $h(\lambda) \leq m$ and an integer $0 \leq s \leq n - l(\lambda) - w(\lambda) + 1$
2. Order: $(\lambda, s) \leq (\mu, t)$ if $\lambda \subseteq \mu$ and $s \geq t$.
3. Inconsistency: $(\lambda, s) \cdots (\mu, t)$ if $\lambda \not\subseteq \mu$ and $\mu \not\subseteq \lambda$.



Theorem 4. The configuration space $\mathcal{S}_n$ of the robotic arm of length n in a tunnel of width 2 is a $\mathrm{CAT}(0)$ cubical complex. Its corresponding PIP is the coral PIP $C_{2,n}$ defined above.

We use the coral PIP $C_{m,n}$ as a remote control for the robot $R_{m,n}$; this allows us to implement an algorithm to move the robotic arm optimally, thus solving Problem 1.

Sketch of proof. The key idea is to encode a position of the robot into a *coral snake tableau*:



and analyze the combinatorics of these tableaux.

Recent results. We have:
- an algorithm, implemented in Python, that moves the robot from one position to another efficiently, solving Problem 1 for a robotic arm in a tunnel of arbitrary width.
- an explicit formula for the distance between any two positions of the robot.
- an explicit formula for the diameter of the transition graph.