LAGRANGIAN GEOMETRY OF MATROIDS

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ABSTRACT. We introduce the conormal fan of a matroid M, which is a Lagrangian analog of the Bergman fan of M. We use the conormal fan to give a Lagrangian interpretation of the Chern–Schwartz–MacPherson cycle of M. This allows us to express the h-vector of the broken circuit complex of M in terms of the intersection theory of the conormal fan of M. We also develop general tools for tropical Hodge theory to prove that the conormal fan satisfies Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann relations. The Lagrangian interpretation of the Chern–Schwartz–MacPherson cycle of M, when combined with the Hodge–Riemann relations for the conormal fan of M, implies Brylawski's and Dawson's conjectures that the h-vectors of the broken circuit complex and the independence complex of M are log-concave sequences.

CONTENTS

1. Introduction

1.1. **Geometry of matroids.** A *matroid* M on a finite set E is a nonempty collection of subsets of E, called *flats* of M, that satisfies the following properties:

(1) The intersection of any two flats is a flat.

(2) For any flat F, any element in $E - F$ is contained in exactly one flat that is minimal among the flats strictly containing F.

The set $\mathcal{L}(\mathrm{M})$ of all flats of M is a geometric lattice, and all geometric lattices arise in this way from a matroid [Wel76, Chapter 3]. The theory of matroids captures the combinatorial essence shared by natural notions of independence in linear algebra, graph theory, matching theory, the theory of field extensions, and the theory of routings, among others.

Gian-Carlo Rota, who helped lay down the foundations of the field, was one of its most energetic ambassadors. He rejected the "ineffably cacophonous" name of matroids, preferring to call them combinatorial geometries instead. This alternative name never really caught on, but the geometric roots of the field have since grown much deeper, bearing many new fruits. The geometric approach to matroid theory has recently led to solutions of long-standing conjectures, and to the development of fascinating mathematics at the intersection of combinatorics, algebra, and geometry.

There are at least three useful polyhedral models of a matroid M. For a short survey, see [Ard18]. The first one is the *basis polytope* of M introduced by Edmonds in optimization and Gelfand–Goresky-MacPherson-Serganova in algebraic geometry. It reveals the intricate relationship of matroids with the Grassmannian variety and the special linear group. The second model is the *Bergman fan* of M, introduced by Sturmfels and Ardila–Klivans in tropical geometry. It was used by Adiprasito–Huh–Katz to prove the log-concavity of the f-vectors of the independence complex and the broken circuit complex of M. The third model, which we call the *conormal fan* of M, is the main character of this paper. We use its intersection-theoretic and Hodge-theoretic properties to prove conjectures of Brylawski [Bry82], Dawson [Daw84], and Swartz [Swa03] on the h-vectors of the independence complex and the broken circuit complex of M.

1.2. **Conormal fans and their geometry.** Throughout the paper, we write $r+1$ for the rank of M, write $n+1$ for the cardinality of E, and suppose that n is positive.[1] Following [MS15], we define the *tropical projective torus* of E to be the n-dimensional vector space

$$\mathrm{N}_E = \mathbb{R}^E/\operatorname{span}(\mathbf{e}_E), \qquad \mathbf{e}_E = \sum_{i\in E} \mathbf{e}_i.$$

The tropical projective torus is equipped with the functions

$$\alpha_j(z) = \max_{i\in E}(z_j - z_i), \text{ one for each element } j \text{ of } E.$$

These functions are equal to each other modulo global linear functions on N_E, and we write α for the common equivalence class of α_j. The *Bergman fan* of M, denoted Σ_{M}, is an r-dimensional fan in the n-dimensional vector space N_E whose underlying set is the *tropical linear space*

$$\operatorname{trop}(\mathrm{M}) = \Big\{ z \,|\, \min_{i\in C}(z_i) \text{ is achieved at least twice for every circuit } C \text{ of } \mathrm{M} \Big\} \subseteq \mathrm{N}_E\,.$$

It is a subfan of the *permutohedral fan* Σ_E cut out by the hyperplanes $x_i = x_j$ for each pair of distinct elements i and j in E. This is the normal fan of the *permutohedron* Π_E. The functions α_j are piecewise linear on the permutohedral fan, and hence piecewise linear on the Bergman fan of M.[2]

Tropical linear spaces are central objects in tropical geometry: For any linear subspace V of $\mathbb{C}^E$, the tropicalization of the intersection of $\mathbb{P}(V)$ with the torus of $\mathbb{P}(\mathbb{C}^E)$ is the tropical linear space of the linear matroid on E represented by V [Stu02]. Furthermore, tropical linear spaces are precisely the tropical fans of degree one with respect to α, that is, the tropical analogs of

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[1] There are exactly two matroids on a single element ground set, the *loop* and the *coloop*, which are dual to each other. These matroids will play a special role in our inductive arguments.

[2] A continuous function f is said to be *piecewise linear* on a fan Σ if the restriction of f to any cone in Σ is linear. In this case, we say that the fan Σ supports the piecewise linear function f.

linear spaces [Fin13]. Tropical manifolds are thus defined to be spaces that locally look like Bergman fans of matroids [IKMZ19].

Adiprasito, Huh, and Katz showed that the Chow ring of the Bergman fan of M satisfies Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann relations [AHK18]. Furthermore, they interpreted the entries of the f-vector of the reduced broken circuit complex of M – an invariant of the matroid generalizing the chromatic polynomial for graphs – as intersection numbers in the Chow ring of Σ_{M}. The geometric interpretation then implied the log-concavity of the coefficients of the *characteristic polynomial* and the *reduced characteristic polynomial*

$$\chi_{\mathrm{M}}(q) \coloneqq \sum_{F \in \mathcal{L}(\mathrm{M})} \mu(\varnothing, F) q^{\mathrm{corank}(F)}, \qquad \overline{\chi}_{\mathrm{M}}(q) \coloneqq \chi_{\mathrm{M}}(q)/(q-1),$$

where μ is the *Möbius function* on the geometric lattice $\mathcal{L}(\mathrm{M})$ for a loopless matroid M.[3]

The conormal fan $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ is an alternative polyhedral model for M. Its construction uses the dual matroid $\mathrm{M}^\perp$, the matroid on E whose bases are the complements of bases of M. We refer to [Oxl11] for background on matroid duality and other general facts on matroids. A central role is played by the *addition map*

$$\mathrm{N}_{E,E} \coloneqq \mathrm{N}_E \oplus \mathrm{N}_E \longrightarrow \mathrm{N}_E, \qquad (z,w) \longmapsto z+w.$$

The function α_j on N_E pulls back to a function δ_j on $\mathrm{N}_{E,E}$ under the addition map. Explicitly,

$$\delta_j(z,w) = \max_{i \in E}(z_j + w_j - z_i - w_i).$$

The function δ_j is piecewise linear on a fan that we construct, called the *bipermutohedral fan* $\Sigma_{E,E}$. This is the normal fan of a convex polytope $\Pi_{E,E}$ that we call the *bipermutohedron*. The functions δ_j for j in E are equal to each other modulo global linear functions on $\mathrm{N}_{E,E}$, and we write δ for their common equivalence class.

The *cotangent fan* Ω_E is the subfan of the bipermutohedral fan $\Sigma_{E,E}$ whose underlying set is the tropical hypersurface

$$\mathrm{trop}(\delta) = \Big\{(z,w) \mid \min_{i \in E}\{z_i + w_i\} \text{ is achieved at least twice}\Big\} \subseteq \mathrm{N}_{E,E}\,.$$

We show the containment

$$\mathrm{trop}(\mathrm{M}) \times \mathrm{trop}(\mathrm{M}^\perp) \subseteq \mathrm{trop}(\delta),$$

and define the *conormal fan* $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ to be the subfan of the cotangent fan Ω_E that subdivides the product $\mathrm{trop}(\mathrm{M}) \times \mathrm{trop}(\mathrm{M}^\perp)$. For our purposes, it is necessary to work with the conormal fan of M instead of the product of the Bergman fans of M and $\mathrm{M}^\perp$, because the function δ_j need not be piecewise linear on the product of the Bergman fans.

[3] If M has a loop, by definition, the characteristic polynomial and the reduced characteristic polynomial of M are zero.

The projections to the summands of $\mathrm{N}_{E,E}$ define morphisms of fans[4]

$$\pi : \Sigma_{\mathrm{M},\mathrm{M}^\perp} \longrightarrow \Sigma_{\mathrm{M}} \quad \text{and} \quad \overline{\pi} : \Sigma_{\mathrm{M},\mathrm{M}^\perp} \longrightarrow \Sigma_{\mathrm{M}^\perp}.$$

Thus, in addition to the functions δ_j, the conormal fan of M supports the pullbacks of α_j in M and $\overline{\alpha}_j$ in $\mathrm{M}^\perp$, which are the piecewise linear functions

$$\gamma_j(z,w) = \max_{i \in E}(z_j - z_i) \quad \text{and} \quad \overline{\gamma}_j(z,w) = \max_{i \in E}(w_j - w_i).$$

These define the equivalence classes γ and $\overline{\gamma}$ of functions on $\mathrm{N}_{E,E}$.

The conormal fan is a tropical analog of the incidence variety appearing in the classical theory of projective duality. For a subvariety X of a projective space $\mathbb{P}$, the incidence variety $\mathcal{I}_X$ is a subvariety of the product of $\mathbb{P}$ with the dual projective space $\mathbb{P}^\vee$ that projects onto X and its dual $X^\vee$. Over the smooth locus of X, the incidence variety $\mathcal{I}_X$ is the total space of the projectivized conormal bundle of X and, over the smooth locus of $X^\vee$, it is the total space of the projectivized conormal bundle of $X^\vee$.[5] We refer to [GKZ94] for a modern exposition of the theory of projective duality.

We use the conormal fan of M to give a geometric interpretation of the polynomial $\overline{\chi}_{\mathrm{M}}(q+1)$, whose coefficients form the h-vector of the broken circuit complex of M with alternating signs. In particular, we give a geometric formula for *Crapo's beta invariant*

$$\beta(\mathrm{M}) \coloneqq (-1)^r\, \overline{\chi}_{\mathrm{M}}(1).$$

This new tropical geometry is inspired by the Lagrangian geometry of conormal varieties in classical algebraic geometry, as we now explain.

Consider the category of complex algebraic varieties with proper morphisms. According to a conjecture of Deligne and Grothendieck, there is a unique natural transformation "csm" from the functor of constructible functions on complex algebraic varieties to the homology of complex algebraic varieties such that, for any smooth variety X,

$$\mathrm{csm}(1_X) = c(TX) \cap [X] = (\text{the total homology Chern class of the tangent bundle of } X).$$

The conjecture was proved by MacPherson [Mac74], and it was recognized later in [BS81] that the class $\mathrm{csm}(1_X)$, for possibly singular X, coincides with a class constructed earlier by Schwartz [Sch65]. For any constructible subset X of Y, the k-th Chern–Schwartz–MacPherson class of X in Y is the homology class

$$\mathrm{csm}_k(1_X) \in H_{2k}(Y).$$

Aiming to introduce a tropical analog of this theory, López de Medrano, Rincón, and Shaw introduced the Chern–Schwartz–MacPherson cycle of the Bergman fan of M in [LdMRS20]: The

[4] A *morphism* from a fan Σ_1 in $\mathrm{N}_1 = \mathbb{R} \otimes \mathrm{N}_{1,\mathbb{Z}}$ to a fan Σ_2 in $\mathrm{N}_2 = \mathbb{R} \otimes \mathrm{N}_{2,\mathbb{Z}}$ is an integral linear map from N_1 to N_2 such that the image of any cone in Σ_1 is a subset of a cone in Σ_2. In the context of toric geometry, a morphism from Σ_1 to Σ_2 can be identified with a toric morphism from the toric variety of Σ_1 to the toric variety of Σ_2 [CLS11, Chapter 3].

[5] Thus, to be precise, the conormal fan is a tropical analog of the projectivized conormal variety and the cotangent fan is a tropical analog of the projectivized cotangent space. We trust that the omission of the term "projectivized" will cause no confusion.

k-th Chern–Schwartz–MacPherson cycle of M is the weighted fan $\mathrm{csm}_k(\mathrm{M})$ supported on the k-dimensional skeleton of Σ_{M} with the weights

$$w(\sigma_{\mathcal{F}}) = (-1)^{r-k}\beta(\mathrm{M}[\mathcal{F}])$$

where $\sigma_{\mathcal{F}}$ is the k-dimensional cone corresponding to a flag of flats $\mathcal{F}$ of M, $\mathrm{M}(i)$ is the minor of M corresponding to the i-th interval in $\mathcal{F}$, and $\beta(\mathrm{M}[\mathcal{F}]) \coloneqq \beta(\mathrm{M}(1)) \cdots \beta(\mathrm{M}(k+1))$ is the *beta invariant of the flag $\mathcal{F}$ in* M. This weighted fan behaves well combinatorially and geometrically. First, the weights satisfy the balancing condition in tropical geometry [LdMRS20, Theorem 1.1], so that we may view the Chern–Schwartz–MacPherson cycle as a Minkowski weight

$$\mathrm{csm}_k(\mathrm{M}) \in \mathrm{MW}_k(\Sigma_{\mathrm{M}}).$$

Second, when $\mathrm{trop}(\mathrm{M})$ is the tropicalization of the intersection $\mathbb{P}(V) \cap (\mathbb{C}^*)^E/\mathbb{C}^*$, the Minkowski weight can be identified with the k-th Chern–Schwartz–MacPherson class of $\mathbb{P}(V) \cap (\mathbb{C}^*)^E/\mathbb{C}^*$ in the toric variety of the permutohedron Π_E [LdMRS20, Theorem 1.2]. Third, the Chern-Schwartz-MacPherson cycles of M satisfy a deletion-contraction formula, a matroid version of the inclusion-exclusion principle [LdMRS20, Proposition 5.2]. It follows that the degrees of these Minkowski weights determine the reduced characteristic polynomial of M by the formula

$$\overline{\chi}_{\mathrm{M}}(q+1) = \sum_{k=0}^{r} \deg(\mathrm{csm}_k(\mathrm{M}))q^k,$$

where the degrees are taken with respect to the class α [LdMRS20, Theorem 1.4]. Fourth, the Chern-Schwartz-MacPherson cycles of matroids can be used to define Chern classes of smooth tropical varieties. In codimension 1, the class agrees with the anticanonical divisor of a tropical variety defined by Mikhalkin in [Mik06]. For smooth tropical surfaces, these classes agree with the Chern classes of tropical surfaces introduced in [Car] and [Sha] to formulate Noether's formula for tropical surfaces.

Schwartz's and MacPherson's constructions of csm are rather subtle. Sabbah later observed that the Chern-Schwartz-MacPherson classes can be interpreted more simply as "shadows" of the characteristic cycles in the cotangent bundle of X. Sabbah summarizes the situation in the following quote from [Sab85]:

> *la théorie des classes de Chern de [Mac74] se ramène à une théorie de Chow sur T^*X, qui ne fait intervenir que des classes fondamentales.*

The functor of constructible functions is replaced with a functor of Lagrangian cycles of T^*X, which are exactly the linear combinations of the *conormal varieties* of the subvarieties of X. In the Lagrangian framework, key operations on constructible functions become more geometric. Similarly, López de Medrano, Rincón, and Shaw's original definition of the Chern–Schwartz–MacPherson cycles of a matroid M is somewhat intricate combinatorially. We prove that they are "shadows" of much simpler cycles under the pushforward map

$$\pi_* : \mathrm{MW}_k(\Sigma_{\mathrm{M},\mathrm{M}^\perp}) \longrightarrow \mathrm{MW}_k(\Sigma_{\mathrm{M}}).$$

See Section 3.1 for a review of basic tropical intersection theory.

Theorem 1.1. When M has no loops and no coloops, for every nonnegative integer $k \leqslant r$,

$$\mathrm{csm}_k(\mathrm{M}) = (-1)^{r-k}\pi_*(\delta^{n-k-1} \cap 1_{\mathrm{M},\mathrm{M}^\perp}),$$

where $1_{\mathrm{M},\mathrm{M}^\perp}$ is the top-dimensional constant Minkowski weight 1 on the conormal fan of M.

It follows from Theorem 1.1 and the projection formula that the reduced characteristic polynomial of M can be expressed in terms of the intersection theory of the conormal fan as follows:

Theorem 1.2. When M has no loops and no coloops, we have

$$\overline{\chi}_{\mathrm{M}}(q+1) = \sum_{k=0}^{r}(-1)^{r-k}\deg(\gamma^k\,\delta^{n-k-1})q^k,$$

where the degrees are taken with respect to the top-dimensional constant Minkowski weight $1_{\mathrm{M},\mathrm{M}^\perp}$ on the conormal fan.

When M is representable over $\mathbb{C}$,[6] the third author gave an algebro-geometric version of Theorem 1.1 in [Huh13]. The complex geometric version of the identity boils down to the fact that the Chern–Schwartz–MacPherson class of a smooth variety X in its normal crossings compactification Y is the total Chern class of the logarithmic tangent bundle:

$$\mathrm{csm}(1_X) = c(\Omega^1_Y(\log Y - X)^\vee) \cap [Y].$$

In fact, the logarithmic formula can be used to construct the natural transformation csm [Alu06]. For precursors of the logarithmic viewpoint, see [Alu99] and [GP02]. The current paper demonstrates that a similar geometry exists for arbitrary M in the tropical setting.

1.3. **Inequalities for matroid invariants.** Let $a_0, a_1, \ldots, a_n$ be a sequence of nonnegative integers, and let d be the largest index with nonzero a_d.

- The sequence is said to be *unimodal* if

$$a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{k-1} \leqslant a_k \geqslant a_{k+1} \geqslant \cdots \geqslant a_n \ \text{ for some } 0 \leqslant k \leqslant n.$$

- The sequence is said to be *log-concave* if

$$a_{k-1}a_{k+1} \leqslant a_k^2 \ \text{ for all } 0 < k < n.$$

- The sequence is said to be *flawless* if

$$a_k \leqslant a_{d-k} \ \text{ for all } 0 \leqslant k \leqslant d/2.$$

Many enumerative sequences are conjectured to have these properties, but proving them often turns out to be difficult. Combinatorialists have been interested in these conjectures because

[6]We say that M is *representable* over a field $\mathbb{F}$ if there exists a linear subspace $V \subseteq \mathbb{F}^E$ such that $S \subseteq E$ is independent in M if and only if the projection from V to $\mathbb{F}^S$ is surjective. Almost all matroids are not representable over any field [Nel18].

their solution typically requires a fundamentally new construction or connection with a distant field, thus revealing hidden structural information about the objects in question. For surveys of known results and open problems, see [Bre94] and [Sta89, Sta00].

A *simplicial complex* Δ is a collection of subsets of a finite set, called *faces* of Δ, that is downward closed. The *face enumerator* of Δ and the *shelling polynomial* of Δ are the polynomials

$$f_\Delta(q) = \sum_{S \in \Delta} q^{|S|} = \sum_{k \geqslant 0} f_k(\Delta) q^k \quad \text{and} \quad h_\Delta(q) = f_\Delta(q-1) = \sum_{k \geqslant 0} h_k(\Delta) q^k.$$

The f*-vector* of a simplicial complex is the sequence of coefficients of its face enumerator, and the h*-vector* of a simplicial complex is the sequence of coefficients of its shelling polynomial. When Δ is *shellable*,[7] the shelling polynomial of Δ enumerates the facets used in shelling Δ, and hence the h-vector of Δ is nonnegative.

We study the f-vectors and h-vectors of the following shellable simplicial complexes associated to M. For a gentle introduction, and for the proof of their shellability, see [Bjö92].

- The *independence complex* $\mathrm{IN}(\mathrm{M})$, the collection of subsets of E that are independent in M.
- The *broken circuit complex* $\mathrm{BC}(\mathrm{M})$, the collection of subsets of E which do not contain any broken circuit of M.

Here a *broken circuit* is a subset obtained from a circuit of M by deleting the least element relative to a fixed ordering of E. The notion was developed by Whitney [Whi32], Rota [Rot64], Wilf [Wil76], and Brylawski [Bry77], for the "chromatic" study of matroids. The f-vector and the h-vector of the broken circuit complex of M are determined by the characteristic polynomial of M, and in particular they do not depend on the chosen ordering of E:

$$\chi_{\mathrm{M}}(q) = \sum_{k=0}^{r+1} (-1)^k f_k(\mathrm{BC}(\mathrm{M})) q^{r-k+1}, \qquad \chi_{\mathrm{M}}(q+1) = \sum_{k=0}^{r+1} (-1)^k h_k(\mathrm{BC}(\mathrm{M})) q^{r-k+1}.$$

Conjecture 1.3. The following holds for any matroid M.

(1) The f-vector of $\mathrm{IN}(\mathrm{M})$ is unimodal, log-concave, and flawless.

(2) The h-vector of $\mathrm{IN}(\mathrm{M})$ is unimodal, log-concave, and flawless.

(3) The f-vector of $\mathrm{BC}(\mathrm{M})$ is unimodal, log-concave, and flawless.

(4) The h-vector of $\mathrm{BC}(\mathrm{M})$ is unimodal, log-concave, and flawless.

Welsh [Wel71] and Mason [Mas72] conjectured the log-concavity of the f-vector of the independence complex.[8] Dawson conjectured the log-concavity of the h-vector of the independence

[7] An r-dimensional pure simplicial complex is said to be shellable if there is an ordering of its facets such that each facet intersects the simplicial complex generated by its predecessors in a pure $(r-1)$-dimensional complex.

[8] In [Mas72], Mason proposed a stronger conjecture that the f-vector of the independence complex of M satisfies

$$\frac{f_k^2}{\binom{n+1}{k}^2} \geqslant \frac{f_{k-1}}{\binom{n+1}{k-1}} \frac{f_{k+1}}{\binom{n+1}{k+1}} \quad \text{for all } k.$$

complex in [Daw84], and independently, Colbourn conjectured the same in [Col87] in the context of network reliability. Hibi conjectured that the h-vector of the independence complex must be flawless [Hib92]. The unimodality and the log-concavity conjectures for the f-vector of the broken circuit complex are due to Heron [Her72], Rota [Rot71], and Welsh [Wel76]. The same conjectures for the chromatic polynomials of graphs were given earlier by Read [Rea68] and Hoggar [Hog74]. We refer to [Whi87, Chapter 8] and [Oxl11, Chapter 15] for overviews and historical accounts. Brylawski [Bry82] conjectured the log-concavity of the h-vector of the broken circuit complex.[9] That the h-vector of the broken circuit complex is flawless stated as an open problem in [Swa03] and reproduced in [JKL18] as a conjecture. We deduce all the above statements using the geometry of conormal fans.

Theorem 1.4. Conjecture 1.3 holds.

We prove the log-concavity of the h-vector of the broken circuit complex using Theorem 1.1. This log-concavity implies all other statements in Conjecture 1.3, thanks to the following known observations:

- For any simplicial complex Δ, the log-concavity of the h-vector implies the log-concavity of the f-vector [Bre94, Corollary 8.4].
- For any pure simplicial complex Δ, the f-vector of Δ is flawless. More generally, any pure O-sequence[10] is flawless [Hib89, Theorem 1.1].
- For any shellable simplicial complex Δ, the h-vector of Δ has no internal zeros, being an O-sequence [Sta77, Theorem 6]. Therefore, if the h-vector of Δ is log-concave, then it is unimodal.
- The broken circuit complex of M is the cone over the *reduced broken circuit complex* of M, and the two simplicial complexes share the same h-vector. The independence complex of M is the reduced broken circuit complex of another matroid, the *free dual extension* of M [Bry77, Theorem 4.2].
- If the h-vector of the broken circuit complex of M is unimodal for all M, then the h-vector of the broken circuit complex of M is flawless for all M [JKL18, Theorem 1.2].

Previous work. The log-concavity of the f-vector of the broken circuit complex was proved in [Huh12] for matroids representable over a field of characteristic 0. The result was extended to

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In [Bry82], Brylawski conjectures the same set of inequalities for the f-vector of the broken circuit complex of M. Mason's stronger conjecture was recently proved in [ALOGV] and [BHa, BHb]. An extension of the same result to matroid quotients was obtained in [EH].

[9] In [Bry82], Brylawski proposed a stronger conjecture that the h-vector of the broken circuit complex of M satisfies

$$\frac{h_k^2}{\binom{n-k}{n-r-1}^2} \geqslant \frac{h_{k-1}}{\binom{n-k+1}{n-r-1}} \frac{h_{k+1}}{\binom{n-k-1}{n-r-1}} \quad \text{for all } k.$$

[10] A sequence of nonnegative integers $h_0, h_1, \dots$ is an *O-sequence* if there is an order ideal of monomials $\mathcal{O}$ such that h_k is the number of degree k monomials in $\mathcal{O}$. The sequence is a *pure O-sequence* if the order ideal $\mathcal{O}$ can be chosen so that all the maximal monomials in $\mathcal{O}$ have the same degree. See [BMMR$^+$12] for a comprehensive survey of pure O-sequences.

matroids representable over some field in [HK12] and to all matroids in [AHK18]. An alternative proof of the same fact using the volume polynomial of a matroid was obtained in [BES]. It was observed in [Len13] that the log-concavity of the f-vector of the broken circuit complex implies that of the independence complex.

For matroids representable over a field of characteristic 0, the log-concavity of the h-vector of the broken circuit complex was proved in [Huh15]. The algebraic geometry behind the log-concavity of the h-vector, which became a model for the Lagrangian geometry of conormal fans in the present paper, was explored in [DGS12] and [Huh13]. In [JKL18], Juhnke-Kubitzke and Le used the result of [Huh15] to deduce that the h-vector of the broken circuit complex is flawless for matroids representable over a field of characteristic 0. The flawlessness of the h-vector of the independence complex was first proved by Chari using a combinatorial decomposition of the independence complex [Cha97]. The result was recovered by Swartz [Swa03] and Hausel [Hau05], who obtained stronger algebraic results. The other cases of Conjecture 1.3 remained open.

Our solution of Conjecture 1.3 was announced in [Ard18]. Very recently, Berget, Spink, and Tseng [BST] have announced an alternative proof of the log-concavity of the h-vector of the independence complex (Dawson's Conjecture 1.3.2). The relationship between our approach and theirs is still to be understood. The h-vector of the broken circuit complex (Brylawski's Conjecture 1.3.4) is not currently accessible through their methods.

1.4. **Tropical Hodge theory.** Let us discuss in more detail the strategy of [AHK18] that led to the log-concavity of the f-vector of the broken circuit complex of M. For the moment, suppose that there is a linear subspace $V \subseteq \mathbb{C}^E$ representing M over $\mathbb{C}$, and consider the variety[11]

$$Y_V = \text{the closure of } \mathbb{P}(V) \cap (\mathbb{C}^*)^E/\mathbb{C}^* \text{ in the toric variety of the permutohedron } X(\Sigma_E).$$

If nonempty, Y_V is an r-dimensional smooth projective complex variety which is, in fact, contained in the torus invariant open subset of $X(\Sigma_E)$ corresponding to the Bergman fan of M:

$$Y_V \subseteq X(\Sigma_{\mathrm{M}}) \subseteq X(\Sigma_E).$$

The work of Feichtner and Yuzvinsky [FY04], which builds up on the work of De Concini and Procesi [DCP95], reveals that the inclusion maps induce isomorphisms between integral cohomology and Chow rings:

$$H^{2\bullet}(Y_V,\mathbb{Z}) \simeq A^\bullet(Y_V,\mathbb{Z}) \simeq A^\bullet(X(\Sigma_{\mathrm{M}}),\mathbb{Z}).$$

As a result, the Chow ring of the n-dimensional variety $X(\Sigma_{\mathrm{M}})$ has the structure of the even part of the cohomology ring of an r-dimensional smooth projective variety. Remarkably, this structure on the Chow ring of $X(\Sigma_{\mathrm{M}})$ persists for any matroid M, even if M does not admit any representation over any field. In particular, the Chow ring of $X(\Sigma_{\mathrm{M}})$ satisfies the Poincaré

[11]Throughout the paper, the toric variety of a fan in N_E refers to the one constructed with respect to the lattice $\mathbb{Z}^E/\mathbb{Z}$. Similarly, the toric variety of a fan in $\mathrm{N}_{E,E}$ refers to the one constructed with respect to the lattice $\mathbb{Z}^E/\mathbb{Z} \oplus \mathbb{Z}^E/\mathbb{Z}$.

duality, the hard Lefschetz theorem, and the Hodge–Riemann relations [AHK18]. For a simpler proof of the three properties of the Chow ring, based on its semi-small decomposition, see [BHM+].

For a simplicial fan Σ, let $A(\Sigma)$ be the ring of real-valued piecewise polynomial functions on Σ modulo the ideal of the linear functions on Σ, and let $\mathcal{K}(\Sigma)$ be the cone of *strictly convex* piecewise linear functions on Σ. We formalize the above properties of the Bergman fan of M as follows.

Definition 1.5. A d-dimensional simplicial fan Σ is *Lefschetz* if it satisfies the following.

(1) (Fundamental weight) The group of d-dimensional Minkowski weights on Σ is generated by a positive Minkowski weight w. We write $\deg$ for the corresponding linear isomorphism

$$\deg : A^d(\Sigma) \longrightarrow \mathbb{R}, \qquad \eta \longmapsto \eta \cap w.$$

(2) (Poincaré duality) For any $0 \leqslant k \leqslant d$, the bilinear map of the multiplication

$$A^k(\Sigma) \times A^{d-k}(\Sigma) \longrightarrow A^d(\Sigma) \xrightarrow{\deg} \mathbb{R}$$

is nondegenerate.

(3) (Hard Lefschetz property) For any $0 \leqslant k \leqslant \frac{d}{2}$ and any $\ell \in \mathcal{K}(\Sigma)$, the multiplication map

$$A^k(\Sigma) \to A^{d-k}(\Sigma), \qquad \eta \longmapsto \ell^{d-2k}\eta$$

is a linear isomorphism.

(4) (Hodge–Riemann relations) For any $0 \leqslant k \leqslant \frac{d}{2}$ and any $\ell \in \mathcal{K}(\Sigma)$, the bilinear form

$$A^k(\Sigma) \times A^k(\Sigma) \longmapsto \mathbb{R}, \qquad (\eta_1, \eta_2) \longmapsto (-1)^k \deg(\ell^{d-2k}\eta_1\eta_2)$$

is positive definite when restricted to the kernel of the multiplication map ℓ^{d-2k+1}.

(5) (Hereditary property) For any $0 < k \leqslant d$ and any k-dimensional cone σ in Σ, the star of σ in Σ is a Lefschetz fan of dimension $d-k$.

The Hodge–Riemann relations give analogs of the Alexandrov–Fenchel inequality amongst degrees of products of convex piecewise linear functions $\ell_1, \ell_2, \ldots, \ell_d$ on Σ:

$$\deg(\ell_1\ell_2\ell_3\cdots\ell_d)^2 \geqslant \deg(\ell_1\ell_1\ell_3\cdots\ell_d)\deg(\ell_2\ell_2\ell_3\cdots\ell_d).$$

The Bergman fan of a matroid M is Lefschetz, and the log-concavity of the f-vector of the broken circuit complex of M follows from the Hodge–Riemann relations for the Bergman fan of M [AHK18].

We establish the log-concavity of the h-vector of the broken circuit complex of M in the same way, using the conormal fan of M in place of the Bergman fan of M. Theorem 1.2 relates the intersection theory of the conormal fan of M to the h-vector of the broken circuit complex of M via the Chern-Schwartz-MacPherson cycles of M. In order to proceed, we need to show that the conormal fan of M is Lefschetz. We obtain this from the following general result.

Theorem 1.6. Let Σ_1 and Σ_2 be simplicial fans that have the same support $|\Sigma_1| = |\Sigma_2|$. If $\mathcal{K}(\Sigma_1)$ and $\mathcal{K}(\Sigma_2)$ are nonempty, then Σ_1 is Lefschetz if and only if Σ_2 is Lefschetz.

Theorem 1.6 implies, for example, that the reduced normal fan of any simple polytope is Lefschetz, because the reduced normal fan of a simplex is Lefschetz.[12] In the context of matroid theory, Theorem 1.6 implies that the conormal fan of M is Lefschetz, because the Bergman fans of M and $\mathrm{M}^\perp$ are Lefschetz and the product of Lefschetz fans is Lefschetz.

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2. The bipermutohedral fan

Let E be a finite set of cardinality $n+1$. For notational convenience, we often identify E with the set of nonnegative integers at most n. As before, we let N_E be the n-dimensional space

$$\mathrm{N}_E = \mathbb{R}^E/\operatorname{span}(\mathbf{e}_E), \qquad \mathbf{e}_E = \sum_{i \in E} \mathbf{e}_i.$$

Let $\mathrm{N}_{E,E}$ be the $2n$-dimensional space $\mathrm{N}_E \oplus \mathrm{N}_E$, and let μ be the addition map

$$\mu : \mathrm{N}_{E,E} \longrightarrow \mathrm{N}_E, \qquad (z,w) \longmapsto z + w.$$

Throughout the paper, all fans in N_E will be rational with respect to the lattice $\mathbb{Z}^E/\mathbb{Z}\mathbf{e}_E$, and all fans in $\mathrm{N}_{E,E}$ will be rational with respect to the lattice $\mathbb{Z}^E/\mathbb{Z}\mathbf{e}_E \oplus \mathbb{Z}^E/\mathbb{Z}\mathbf{e}_E$. We follow [CLS11] when using the terms *fan* and *generalized fan*: A generalized fan is a fan if and only if each of its cone is strongly convex. The notion of morphism of fans is extended to morphism of generalized fans in the obvious way. For any subset S of E, we write $\mathbf{e}_S$ and $\mathbf{f}_S$ for the vectors

$$\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i, \qquad \mathbf{f}_S = \sum_{i \in S} \mathbf{f}_i,$$

where $\mathbf{e}_i$ are the standard basis vectors of $\mathbb{R}^E$ defining the first summand of $\mathrm{N}_{E,E}$ and $\mathbf{f}_i$ are the standard basis vectors of $\mathbb{R}^E$ defining the second summand of $\mathrm{N}_{E,E}$.

In this section, we construct a complete simplicial fan $\Sigma_{E,E}$ in $\mathrm{N}_{E,E}$ that will play a central role in this paper. We offer five equivalent descriptions; each one of them will play an important

[12] McMullen gave an elementary proof of this fact in [McM93]. See [Tim99] and [FK10] for alternative presentations. Our proof of Theorem 1.6 is modeled on these arguments. Theorem 1.6 gives another proof of the necessity of McMullen's bounds [McM93] on the face numbers of simplicial polytopes.

role for us. We call it the *bipermutohedral fan* because it is the normal fan of a polytope which we call the *bipermutohedron*. Before we begin defining the bipermutohedral fan $\Sigma_{E,E}$ in $\mathrm{N}_{E,E}$, we recall some basic facts on the permutohedral fan Σ_E in N_E.

2.1. **The normal fan of the simplex.** Consider the standard n-dimensional simplex

$$\mathrm{conv}\{\mathbf{e}_i\}_{i \in E} \subseteq \mathbb{R}^E.$$

Its normal fan in $\mathbb{R}^E$ has the lineality space spanned by $\mathbf{e}_E$. For any convex polytope, we call the quotient of the normal fan by its lineality space the *reduced normal fan* of the polytope.[13] For example, the reduced normal fan of the standard simplex, denoted Γ_E, is the complete fan in N_E with the cones

$$\sigma_S \coloneqq \mathrm{cone}\{\mathbf{e}_i\}_{i \in S} \subseteq \mathrm{N}_E, \ \text{ for every proper subset } S \text{ of } E.$$

The cone σ_S consists of the points $z \in \mathrm{N}_E$ such that $\min\limits_{i \in E} z_i = z_s$ for all s not in S. For each element j of E, the function $\alpha_j = \max\limits_{i \in E}\{z_j - z_i\}$ is piecewise linear on the fan Γ_E. These piecewise linear functions are equal to each other modulo global linear functions on N_E, and we write α for the common equivalence class of α_j.

2.2. **The normal fan of the permutohedron.** Let Π_E be the n-dimensional permutohedron

$$\mathrm{conv}\Big\{(x_0, x_1, \dots, x_n) \mid x_0, x_1, \dots, x_n \text{ is a permutation of } 0, 1, \dots, n\Big\} \subseteq \mathbb{R}^E.$$

The *permutohedral fan* Σ_E, also known as the *braid fan* or the *type A Coxeter complex*, is the reduced normal fan of the permutohedron Π_E. It is the complete simplicial fan in N_E whose chambers are separated by the n-dimensional *braid arrangement*, the real hyperplane arrangement in N_E consisting of the $\binom{n+1}{2}$ hyperplanes

$$z_i = z_j, \ \text{ for distinct elements } i \text{ and } j \text{ of } E.$$

The face of the permutohedral fan containing a given point z in its relative interior is determined by the relative order of its homogeneous coordinates $(z_0, \dots, z_n)$. Therefore, the faces of the permutohedral fan correspond to the ordered set partitions

$$\mathcal{P} = (E = P_1 \sqcup \cdots \sqcup P_{k+1}),$$

which are in bijection with the strictly increasing sequences of nonempty proper subsets

$$\mathcal{S} = (\varnothing \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E), \qquad S_m = \bigcup_{\ell=1}^{m} P_\ell.$$

The collection of ordered set partitions of E form a poset under *adjacent refinement*, where $\mathcal{P} \leqslant \mathcal{P}'$ if $\mathcal{P}$ can be obtained from $\mathcal{P}'$ by merging adjacent parts.

---

[13]The normal fan of a convex polytope P in a vector space is a generalized fan in the dual space whose face poset is anti-isomorphic to the face poset of P. Unlike the reduced normal fan, the normal fan of a polytope is a generalized fan, and need not be a fan. We trust that the use of the term “normal fan” will cause no confusion.

Proposition 2.1. The face poset of the permutohedral fan Σ_E is isomorphic to the poset of ordered set partitions of E.

Thus the permutohedral fan has $2(2^n - 1)$ rays corresponding to the nonempty proper subsets of E and $(n+1)!$ chambers corresponding to the permutations of E.

We now describe the permutohedral fan in terms of its rays. Two subsets S and S' of E are said to be *comparable* if

$$S \subseteq S' \text{ or } S \supseteq S'.$$

A *flag* in E is a set of pairwise comparable subsets of E. For any flag $\mathcal{S}$ of subsets of E, we define

$$\sigma_{\mathcal{S}} = \text{cone}\{\mathbf{e}_S\}_{S \in \mathcal{S}} \subseteq \mathrm{N}_E\,.$$

We identify a flag in E with the strictly increasing sequence obtained by ordering the subsets in the flag.

Proposition 2.2. The permutohedral fan Σ_E is the complete fan in N_E with the cones

$$\sigma_{\mathcal{S}} = \text{cone}\{\mathbf{e}_S\}_{S \in \mathcal{S}}, \quad \text{where } \mathcal{S} \text{ is a flag of nonempty proper subsets of } E.$$

For example, the cone corresponding to the ordered set partition $25|013|4$ is

$$\text{cone}(\mathbf{e}_{25}, \mathbf{e}_{01235}) = \{z \in \mathrm{N}_E \mid z_2 = z_5 \geqslant z_0 = z_1 = z_3 \geqslant z_4\}.$$

Proposition 2.2 shows that the permutohedral fan is a *unimodular fan*: The set of primitive ray generators in any cone in Σ_E is a subset of a basis of the free abelian group $\mathbb{Z}^E/\mathbb{Z}$. It also shows that the permutohedral fan is a refinement of the fan Γ_E in Section 2.1.

It will be useful to view the permutohedral fan as a configuration space as follows. Regard N_E as the space of E-tuples of points $(p_0, \ldots, p_n)$ moving in the real line, modulo simultaneous translation:

$$p = (p_0, \ldots, p_n) = (p_0 + \lambda, \ldots, p_n + \lambda) \text{ for any } \lambda \in \mathbb{R}.$$

The *ordered set partition of* p, denoted $\pi(p)$, is obtained by reading the labels of the points in the real line from right to left, as shown in Figure 1. This model gives the permutohedral fan Σ_E the following geometric interpretation.

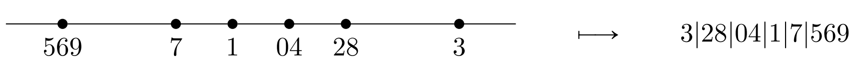


FIGURE 1. An E-tuple of points p and its ordered set partition $\pi(p) = 3|28|04|1|7|569$.

Proposition 2.3. The permutohedral fan Σ_E is the configuration space of E-tuples of points in the real line modulo simultaneous translation, stratified according to their ordered set partition.

In Section 2.4, we give an analogous description of the bipermutohedral fan $\Sigma_{E,E}$ as a configuration space of E-tuples of points in the real plane.

2.3. **The bipermutohedral fan as a subdivision.** Denote a point in $\mathrm{N}_{E,E}$ by (z, w). We construct the bipermutohedral fan $\Sigma_{E,E}$ in $\mathrm{N}_{E,E}$ as follows.

First, we subdivide $\mathrm{N}_{E,E}$ into the *charts* $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_n$, where $\mathcal{C}_k$ is the cone

$$\mathcal{C}_k = \Big\{(z, w) \mid \min_{i \in E}(z_i + w_i) = z_k + w_k\Big\}.$$

These form the chambers of a complete generalized fan in $\mathrm{N}_{E,E}$, denoted Δ_E. The chamber $\mathcal{C}_k$ is the inverse image of the cone σ_{E-k} under the addition map, and hence Δ_E is the coarsest complete generalized fan in $\mathrm{N}_{E,E}$ for which the addition map is a morphism to the fan Γ_E in Section 2.1. To each chart $\mathcal{C}_k$, we associate the linear functions

$$Z_i = z_i - z_k, \quad W_i = -w_i + w_k, \ \text{ for every } i \text{ in } E.$$

Omitting the zero function $Z_k = W_k$, we obtain a coordinate system (Z, W) for $\mathrm{N}_{E,E}$ such that

$$\mathcal{C}_k = \Big\{(Z, W) \mid Z_i \geqslant W_i \text{ for every } i \text{ in } E\Big\}.$$

This coordinate system depends on k, but we will drop k from the notation for better readability.

Second, we consider the subdivision Σ_k of the cone $\mathcal{C}_k$ obtained from the braid arrangement of $\binom{2n+1}{2}$ hyperplanes

$$Z_a = Z_b, \quad W_a = W_b, \quad Z_a = W_b, \quad \text{for all } a \text{ and } b \text{ in } E.$$

Note that the arrangement contains the n hyperplanes that cut out $\mathcal{C}_k$ in $\mathrm{N}_{E,E}$. One may view the subdivision Σ_k of $\mathcal{C}_k$ as a copy of $1/2^n$-th of the $2n$-dimensional permutohedral fan.

Proposition 2.4. The union of the fans Σ_i for $i \in E$ is a fan in $N_{E,E}$. We call it the *bipermutohedral fan* $\Sigma_{E,E}$.

Proof. To check that $\Sigma_{E,E}$ is indeed a fan, we need to check that the fans Σ_i glue compatibly along the boundaries of $\mathcal{C}_i$. For this, we verify that Σ_i and Σ_j induce the same subdivision on $\mathcal{C}_i \cap \mathcal{C}_j$ for all $i \neq j$.

Consider the system of linear functions (Z, W) for $\mathcal{C}_i$ and the system of linear functions (Z', W') for $\mathcal{C}_j$. It is straightforward to check that, for any point in $\mathrm{N}_{E,E}$, we have

$$Z_a - Z_b = Z'_a - Z'_b \ \text{ and } \ W_a - W_b = W'_a - W'_b \ \text{ for all } a \text{ and } b \text{ in } E.$$

Furthermore, on the intersection of $\mathcal{C}_i$ and $\mathcal{C}_j$, where $z_i + w_i = z_j + w_j$, we have

$$Z_a - W_b = (z_a - z_i) - (w_i - w_b) = (z_a - z_j) - (w_j - w_b) = Z'_a - W'_b.$$

Thus the hyperplanes separating the chambers of Σ_i and Σ_j have the same intersections with $\mathcal{C}_i \cap \mathcal{C}_j$. $\square$

The following subfan of the bipermutohedral fan will serve as a guide toward Theorem 1.1.

Definition 2.5. The *cotangent fan* Ω_E is the union of the fans $\Sigma_i \cap \Sigma_j$ for $i \neq j \in E$.

In other words, Ω_E is the subfan of $\Sigma_{E,E}$ whose support is the tropical hypersurface

$$\mathrm{trop}(\delta) = \Big\{(z,w) \mid \min_{i \in E}(z_i + w_i) \text{ is achieved at least twice}\Big\} \subseteq \mathrm{N}_{E,E}\,.$$

In Section 3.4, we show that the cotangent fan contains the conormal fan of any matroid on E.

2.4. **The bipermutohedral fan as a configuration space.** It will be useful to view the bipermutohedral fan $\Sigma_{E,E}$ as a configuration space as follows. Regard $\mathrm{N}_{E,E}$ as the space of E-tuples of points $(p_0, \ldots, p_n)$ moving in the real plane, modulo simultaneous translation:

$$(p_0, \ldots, p_n) = (p_0 + \lambda, \ldots, p_n + \lambda) \text{ for any } \lambda \in \mathbb{R}^2.$$

The point (z,w) in $\mathrm{N}_{E,E}$ corresponds to the points $p_i = (z_i, w_i)$ in $\mathbb{R}^2$ for i in E.

Definition 2.6. A *bisequence* on E is a sequence $\mathcal{B}$ of nonempty subsets of E, called the *parts* of $\mathcal{B}$, such that

(1) every element of E appears in at least one part of $\mathcal{B}$,

(2) every element of E appears in at most two parts of $\mathcal{B}$, and

(3) some element of E appears in exactly one part of $\mathcal{B}$.

The *trivial bisequence* on E is the bisequence with exactly one part E. A *bisubset* of E is a nontrivial bisequence on E of minimal length 2. A *bipermutation* of E is a bisequence on E of maximal length $2n+1$.

We will write bisequences by listing the elements of its parts, separated by vertical bars. For example, the bisequence $\{2\}, \{0,1\}, \{1\}, \{2\}$ on $\{0,1,2\}$ will be written $2|01|1|2$.

Definition 2.7. Let $p = (p_0, \ldots, p_n)$ be an E-tuple of points in $\mathbb{R}^2$.

(1) The *supporting line* of p, denoted $\ell(p)$, is the lowest line of slope -1 containing a point in p.

(2) For each point p_i, the vertical and horizontal projections of p_i onto $\ell(p)$ will be labelled i.

(3) The *bisequence of* p, denoted $\mathcal{B}(p)$, is obtained by reading the labels on $\ell(p)$ from right to left.

See Figure 2 for an illustration of Definition 2.7.

Remark 2.8. One can recover any configuration p from their projections onto the supporting line $\ell(p)$ and their labels. Therefore, modulo translations, we may also consider p as a configuration of $2n+2$ points on the real line labeled $0, 0, 1, 1, \ldots, n, n$ such that at least one pair of points with the same label coincide. This is illustrated at the bottom of Figure 2.

This model gives the bipermutohedral fan $\Sigma_{E,E}$ the following geometric interpretation.

Proposition 2.9. The bipermutohedral fan $\Sigma_{E,E}$ is the configuration space of E-tuples of points in the real plane modulo simultaneous translation, stratified according to their bisequence.

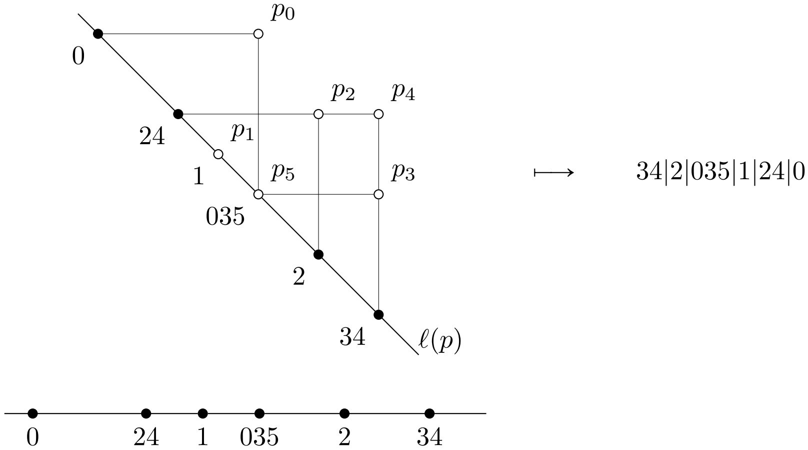


FIGURE 2. An E-tuple of points $p = (p_0, \ldots, p_5)$ in the plane, their vertical and horizontal projections onto the supporting line $\ell(p)$, and the bisequence $\mathcal{B}(p)$.

Proof. Consider a point (z, w) in $\mathrm{N}_{E,E}$ and the associated configuration of points p_i in the plane. The chart $\mathcal{C}_k$ consists of configurations p where k appears exactly once in the bisequence $\mathcal{B}(p)$. In other words, p is in $\mathcal{C}_k$ if and only if p_k is on the supporting line $\ell(p)$. We consider the system of linear functions (Z, W) for $\mathcal{C}_k$ discussed in Section 2.3. The cones in the subdivision Σ_k of $\mathcal{C}_k$ encode the relative order of $Z_0, \ldots, Z_n, W_0, \ldots, W_n$, where

$$Z_k = W_k = 0 \ \text{ and } \ Z_i \geqslant W_i \ \text{ for every } i \text{ in } E.$$

On the other hand, the bisequence $\mathcal{B}(p)$ keeps track of the relative order of the vertical and horizontal projections of p_i onto $\ell(p)$. As shown in Figure 3, after the translation by $(-z_k, -w_k)$, the vertical and horizontal projections of p_i onto $\ell(p)$ are

$$(z_i, z_k + w_k - z_i) - (z_k, w_k) = (Z_i, -Z_i) \ \text{ and } \ (z_k + w_k - w_i, w_i) - (z_k, w_k) = (W_i, -W_i).$$

Their relative order along $\ell(p)$ is given by the relative order of $Z_0, \ldots, Z_n, W_0, \ldots, W_n$. $\square$

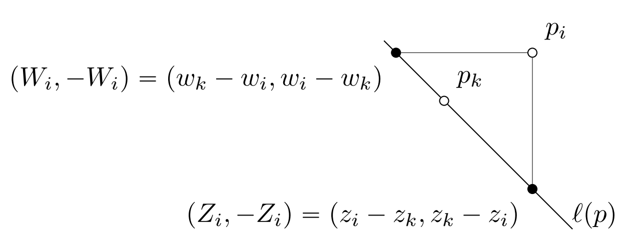


FIGURE 3. The vertical and horizontal projections of p_i onto the supporting line $\ell(p)$, after the translation by $(-z_k, -w_k)$.

The collection of bisequences on E form a poset under *adjacent refinement*, where $\mathcal{B} \leqslant \mathcal{B}'$ if $\mathcal{B}$ can be obtained from $\mathcal{B}'$ by merging adjacent parts. The poset of bisequences on E is a graded poset. Its k-th level consists of the bisequences of $k+1$ nonempty subsets of E, and the top level consists of the bipermutations of E.

Proposition 2.10. The face poset of the bipermutohedral fan $\Sigma_{E,E}$ is isomorphic to the poset of bisequences on E.

Proof. Remark 2.8 shows that, given any bisequence $\mathcal{B}$ on E, there is a configuration p with $\mathcal{B}(p) = \mathcal{B}$. Thus, by Proposition 2.9, the cones in $\Sigma_{E,E}$ are in bijection with the bisequences on E. If a configuration p moves into more special position, then some adjacent parts of $\mathcal{B}(p)$ merge. $\square$

For a bisequence $\mathcal{B}$ on E, we write $\sigma_{\mathcal{B}}$ for the corresponding cone defined by

$$\sigma_{\mathcal{B}} = \text{closure}\Big\{\text{configurations } p \text{ satisfying } \mathcal{B}(p) = \mathcal{B}\Big\} \subseteq \mathrm{N}_{E,E}\,.$$

In terms of the cones $\sigma_{\mathcal{B}}$, the fan Σ_i subdividing the chart $\mathcal{C}_i$ can be described as the subfan

$$\Sigma_i = \{\sigma_{\mathcal{B}} \mid i \text{ appears exactly once in the bisequence } \mathcal{B}\} \subseteq \Sigma_{E,E}.$$

See Figure 4 for an illustration of Proposition 2.10 when $n = 1$.

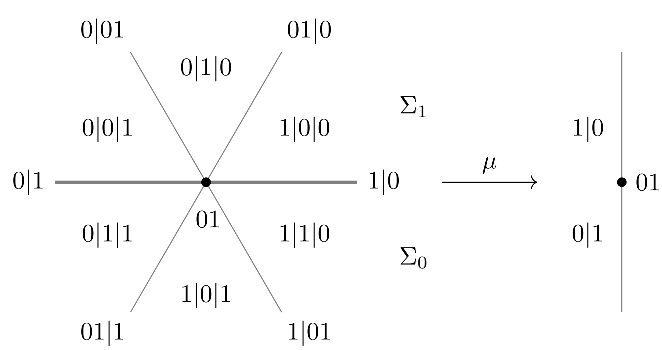


FIGURE 4. The map $\mu\colon \Sigma_{\{0,1\},\{0,1\}} \to \Sigma_{\{0,1\}}$ from the bipermutohedral fan to the permutohedral fan, and the labelling of their cones with bisequences on $\{0,1\}$ and ordered set partitions on $\{0,1\}$, respectively.

2.5. **The bipermutohedral fan as a common refinement.** The importance of the bipermutohedral fan $\Sigma_{E,E}$ stems from its relationship with the normal fan Γ_E of the standard simplex and the permutohedral fan Σ_E described in Sections in Section 2.1 and 2.2. Recall that a *morphism* from a fan Σ_1 in N_1 to a fan Σ_2 in N_2 is an integral linear map from N_1 to N_2 that maps any cone in Σ_1 into a cone in Σ_2.

Proposition 2.11. The bipermutohedral fan $\Sigma_{E,E}$ has the following properties.

(1) The projections $\pi(z,w)=z$ and $\overline{\pi}(z,w)=w$ are morphisms of fans from $\Sigma_{E,E}$ to Σ_E.

(2) The addition map $\mu(z,w)=z+w$ is a morphism of fans from $\Sigma_{E,E}$ to Γ_E.

Proof. That $\Sigma_{E,E}$ has the stated properties follows from the interpretation of Σ_E and $\Sigma_{E,E}$ as configuration spaces, as we now explain. Suppose (z,w) is a point in $\mathrm{N}_{E,E}$ and p is the corresponding E-tuple of points in $\mathbb{R}^2$ modulo simultaneous translation, with corresponding bisequence $\mathcal{B}(p)$. Then the smallest cone of Γ_E containing $z+w$ is given by the entries that appear twice in $\mathcal{B}(p)$. The ordered set partition of z in N_E is given by the first occurrence of each i in $\mathcal{B}(p)$. Similarly, the ordered set partition of w in N_E is given by the order of the last occurrence of each i in $\mathcal{B}(p)$. For example, if a point (z,w) has the bisequence $34|2|035|1|24|0$, as in Figure 2, then the sum $z+w$ is in the cone of 0234 in Γ_E, the first projection z is in the cone of $34|2|05|1$ in Σ_E, and the second projection w is in the cone of $0|24|1|35$ in Σ_E. □

2.6. **The bipermutohedral fan in terms of its rays and cones.** The rays of the bipermutohedral fan $\Sigma_{E,E}$ correspond to the bisubsets of E. In other words, the rays of $\Sigma_{E,E}$ correspond to the ordered pairs of nonempty subsets $S|T$ of E such that

$$S\cup T=E \ \text{ and } \ S\cap T\neq E.$$

Proposition 2.12. The $3(3^n-1)$ rays of the bipermutohedral fan $\Sigma_{E,E}$ are generated by

$$\mathbf{e}_{S|T}\coloneqq \mathbf{e}_S+\mathbf{f}_T, \ \text{ where } S|T \text{ is a bisubset of } E.$$

Proof. The configuration p corresponding to $\mathbf{e}_{S|T}$ has points with labels in $S\cap T$ located at $(1,1)$, the points with labels in $S-T$ located at $(1,0)$, and the points with labels in $T-S$ located at $(0,1)$. The bisequence of p is indeed $S|T$, and hence the conclusion follows from Proposition 2.9. □

Proposition 2.13. The bipermutohedral fan $\Sigma_{E,E}$ has $(2n+2)!/2^{n+1}$ chambers.

Proof. By Proposition 2.10, the chambers correspond to the bipermutations. These are obtained bijectively from the $(2n+2)!/2^{n+1}$ permutations of the multiset $\{0,0,\ldots,n,n\}$ by dropping the last letter in the one-line notation for permutations. For example, the bipermutation $1|0|1|2|3|0|3$ correspond to the permutation 10123032 of $\{0,0,1,1,2,2,3,3\}$. □

It is worth understanding Proposition 2.13 in a different way. Recall that the bipermutohedral fan is obtained by gluing copies of $1/2^n$-th of the $2n$-dimensional permutohedral fan. There are $(n+1)$ such copies, and each copy contains $(2n+1)!/2^n$ chambers, producing the total of $(2n+2)!/2^{n+1}$ chambers. This viewpoint explains why Figure 4 deceivingly looks like a permutohedral fan: For $n=1$, the bipermutohedral fan consists of two glued copies of half of the permutohedral fan.

We now describe the cones in the bipermutohedral fan in terms of their generating rays. Let $\mathcal{B}=B_0|B_1|\cdots|B_k$ be a bisequence on E. Propositions 2.10 and 2.12 show that the rays of the

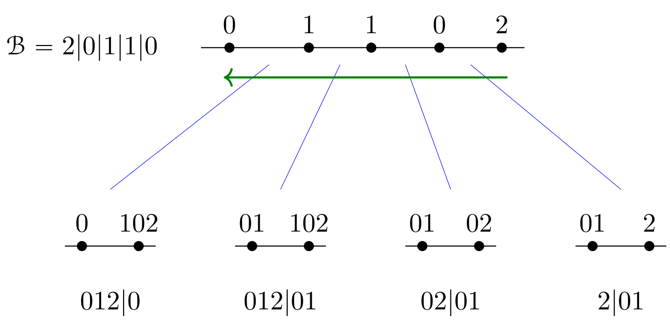
k-dimensional cone $\sigma_{\mathcal{B}}$ are generated by the vectors

$$\mathbf{e}_{S_1|T_1}, \ldots, \mathbf{e}_{S_k|T_k}, \quad \text{where } S_i = \bigcup_{j=0}^{i-1} B_j \text{ and } T_i = \bigcup_{j=i}^{k} B_j.$$

See Figure 5 for an illustration. We use the following table to record the rays of $\sigma_{\mathcal{B}}$:

$\varnothing$	$\subsetneq$	S_1	$\subseteq$	S_2	$\subseteq$	$\cdots$	$\subseteq$	S_k	$\subseteq$	E
E	$\supseteq$	T_1	$\supseteq$	T_2	$\supseteq$	$\cdots$	$\supseteq$	T_k	$\supsetneq$	$\varnothing$

For each index j such that $S_j \subsetneq S_{j+1}$ and $T_j \supsetneq T_{j+1}$, we mark those two strict inclusions in blue. We write $\mathcal{S}(\mathcal{B})|\mathcal{T}(\mathcal{B})$ for the collection of bisubsets $S_i|T_i$ constructed from $\mathcal{B}$ as above by merging adjacent parts. For convenience, we also refer to the pairs $S_0|T_0 = \varnothing|E$ and $S_{k+1}|T_{k+1} = E|\varnothing$.



$\varnothing$	$\subsetneq$	2	$\subseteq$	02	$\subseteq$	012	$\subseteq$	012	$\subseteq$	E
E	$\supseteq$	01	$\supseteq$	01	$\supseteq$	01	$\supseteq$	0	$\supsetneq$	$\varnothing$

FIGURE 5. The cone of $2|0|1|1|0$ has the rays generated by $\mathbf{e}_{2|01}, \mathbf{e}_{02|01}, \mathbf{e}_{012|01}, \mathbf{e}_{012|0}$.

Conversely, we may ask which subsets of k rays in $\Sigma_{E,E}$ generate a k-dimensional cone in $\Sigma_{E,E}$. To answer this question, we introduce the notion of a flag of bisubsets.

Definition 2.14. We say that two bisubsets $S|T$ and $S'|T'$ of E are *comparable* if

$$(S \subseteq S' \text{ and } T \supseteq T') \text{ or } (S \supseteq S' \text{ and } T \subseteq T').$$

A *flag of bisubsets* in E, or a *biflag* in E, is a set $\mathcal{S}|\mathcal{T}$ of pairwise comparable bisubsets of E satisfying

$$\bigcup_{S|T \in \mathcal{S}|\mathcal{T}} S \cap T \neq E.$$

The *length* of a biflag is the number of bisubsets in it.

We have the following useful alternative characterization of biflags in E.

Proposition 2.15. Let $\mathcal{S}$ be an increasing sequence of k nonempty subsets of E, say

$$\mathcal{S} = (\varnothing \subsetneq S_1 \subseteq \cdots \subseteq S_k \subseteq E),$$

and let $\mathcal{T}$ be a decreasing sequence of k nonempty subsets of E, say

$$\mathcal{T} = (E \supseteq T_1 \supseteq \cdots \supseteq T_k \supsetneq \varnothing).$$

Then the set $\mathcal{S}|\mathcal{T}$ consisting of the pairs $S_1|T_1, \ldots, S_k|T_k$ is a flag of bisubsets if and only if

$$S_j \cup T_j = E \text{ for every } 1 \leqslant j \leqslant k \text{ and } S_j \cup T_{j+1} \neq E \text{ for some } 0 \leqslant j \leqslant k.$$

Proof. If $\mathcal{S}|\mathcal{T}$ is a biflag in E, then each $S_j|T_j$ is a bisubset of E, and hence $S_j \cup T_j = E$ for all j. Now let e be an element not in the union of all $S_j \cap T_j$, and consider the largest index i for which $e \notin S_i$. Then $e \in S_{i+1}$, which implies $e \notin T_{i+1}$ by the definition of e. Therefore, $S_i \cup T_{i+1} \neq E$.

Conversely, if $\mathcal{S}$ and $\mathcal{T}$ satisfy the stated conditions, then the pairs $S_j|T_j$ form a set of pairwise comparable bisubsets of E. If e is an element not in $S_j \cup T_{j+1}$ for some index j, then e is not in S_k for all indices $k \leqslant j$ and e is not in T_k for all indices $k > j$. Therefore, e is not in the union of all $S_k \cap T_k$, as desired. □

Note that $S_j \cup T_{j+1} \neq E$ implies that $S_j \subsetneq S_{j+1}$ and $T_j \supsetneq T_{j+1}$, so the table of any biflag has at least one pair of strict inclusions marked in blue.

For a biflag $\mathcal{S}|\mathcal{T}$ of length k, we write $\mathcal{S}$ for the increasing sequence of k nonempty subsets

$$\mathcal{S} = (\varnothing \subsetneq S_1 \subseteq \cdots \subseteq S_k \subseteq E), \text{ where } S_j \text{ are the first parts of the bisubsets in } \mathcal{S}|\mathcal{T},$$

and write $\mathcal{T}$ for the decreasing sequence of k nonempty subsets

$$\mathcal{T} = (E \supseteq T_1 \supseteq \cdots \supseteq T_k \supsetneq \varnothing), \text{ where } T_j \text{ are the second parts of the bisubsets in } \mathcal{S}|\mathcal{T}.$$

We use $\mathcal{S}$ and $\mathcal{T}$ to define $\mathcal{B}(\mathcal{S}|\mathcal{T})$ as the sequence of $k+1$ nonempty sets

$$B_0|B_1|\cdots|B_k, \text{ where } B_j = (S_{j+1} - S_j) \cup (T_j - T_{j+1}).$$

The above construction is an isomorphism between the poset of bisequences under adjacent refinement and the poset of biflags under inclusion.

Proposition 2.16. The bisequences on E are in bijection with the biflags in E. More precisely,

(1) if $\mathcal{B}$ is a bisequence on E, then $\mathcal{S}(\mathcal{B})|\mathcal{T}(\mathcal{B})$ is a biflag in E,

(2) if $\mathcal{S}|\mathcal{T}$ is a biflag in E, then $\mathcal{B}(\mathcal{S}|\mathcal{T})$ is a bisequence on E, and

(3) the constructions $\mathcal{S}(\mathcal{B})|\mathcal{T}(\mathcal{B})$ and $\mathcal{B}(\mathcal{S}|\mathcal{T})$ are inverses to each other.

Note that a bisubset $S|T$ corresponds to the biflag $\{S|T\}$ under the above bijection. For simplicity, we use the two symbols interchangeably.

Proof. (1) Since every element of E appears at least once in the bisequence $\mathcal{B}$, the increasing flag $\mathcal{S}(\mathcal{B})$ and the decreasing flag $\mathcal{T}(\mathcal{B})$ satisfy $S_j \cup T_j = E$ for all j. In addition, since some element of E appears exactly once in $\mathcal{B}$, say in B_j, we have $S_j \cup T_{j+1} \neq E$ for some j. Therefore, by Proposition 2.15, the pair $\mathcal{S}(\mathcal{B})|\mathcal{T}(\mathcal{B})$ is a biflag in E.

(2) Conversely, suppose that $\mathcal{S}|\mathcal{T}$ is a biflag in E. Since $S_1|T_1, \ldots, S_k|T_k$ are pairwise distinct, B_j must be nonempty for all j. Clearly, every element in E must appear in B_j for some j. In addition, each element e in E can occur at most twice in $\mathcal{B}(\mathcal{S}|\mathcal{T})$, namely, in the parts B_a and B_b whose indices satisfy $e \in S_{a+1} - S_a$ and $e \in T_b - T_{b+1}$. Furthermore, by Proposition 2.15, there is an element e not in $S_c \cup T_{c+1}$ for some index c, and in this case we must have $a = b = c$. That element e can occur only in the part B_a of $\mathcal{B}(\mathcal{S}|\mathcal{T})$, and hence $\mathcal{B}(\mathcal{S}|\mathcal{T})$ is indeed a bisequence.

(3) It is straightforward to check that the constructions $\mathcal{S}(\mathcal{B})|\mathcal{T}(\mathcal{B})$ and $\mathcal{B}(\mathcal{S}|\mathcal{T})$ are inverses to each other. □

We identify a biflag $\mathcal{S}|\mathcal{T}$ in E with the sequence of bisubsets of E obtained by ordering the bisubsets in $\mathcal{S}|\mathcal{T}$ as above. For any sequence $\mathcal{S}|\mathcal{T}$ of bisubsets of E, we define

$$\sigma_{\mathcal{S}|\mathcal{T}} = \operatorname{cone}\{\mathbf{e}_{S|T}\}_{S|T \in \mathcal{S}|\mathcal{T}} \subseteq \mathrm{N}_{E,E}\,.$$

Thus, for any bisequence $\mathcal{B}$ on E, we have $\sigma_{\mathcal{B}} = \sigma_{\mathcal{S}(\mathcal{B})|\mathcal{T}(\mathcal{B})}$.

Corollary 2.17. The bipermutohedral fan $\Sigma_{E,E}$ is the complete fan in $\mathrm{N}_{E,E}$ with the cones

$$\sigma_{\mathcal{S}|\mathcal{T}} = \operatorname{cone}\{\mathbf{e}_{S|T}\}_{S|T \in \mathcal{S}|\mathcal{T}}, \quad \text{where } \mathcal{S}|\mathcal{T} \text{ is a flag of bisubsets of } E.$$

Proof. The statement is straightforward, given Propositions 2.10 and 2.16. □

Corollary 2.17 can be used to show that the bipermutohedral fan is a unimodular fan.[14]

Proposition 2.18. The set of primitive ray generators of any chamber of $\Sigma_{E,E}$ is a basis of the free abelian group $\mathbb{Z}^E/\mathbb{Z}\mathbf{e}_E \oplus \mathbb{Z}^E/\mathbb{Z}\mathbf{f}_E$.

Proof. Let $\mathcal{S} = \mathcal{S}(\mathcal{B})$ and $\mathcal{T} = \mathcal{T}(\mathcal{B})$ for a bipermutation $\mathcal{B}$ of E. If 0 is the unique element of E that appears exactly once in $\mathcal{B}$, then

$$\Big\{\mathbf{e}_{S_{j+1}|T_{j+1}} - \mathbf{e}_{S_j|T_j} \mid 0 \text{ is contained in } S_j \cup T_{j+1}\Big\} = \Big\{\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{f}_1, \ldots, \mathbf{f}_n\Big\}.$$

Therefore, the set of $2n$ primitive ray generators of $\sigma_{\mathcal{B}}$ generates $\mathbb{Z}^E/\mathbb{Z}\mathbf{e}_E \oplus \mathbb{Z}^E/\mathbb{Z}\mathbf{f}_E$. □

2.7. **The bipermutohedral fan as the normal fan of the bipermutohedron.** In this section we construct a polytope $\Pi_{E,E}$, called the *bipermutohedron*, whose reduced normal fan is $\Sigma_{E,E}$. We begin by identifying each permutation of the multiset $E \cup \overline{E} := \{0, \overline{0}, 1, \overline{1}, \ldots, n.\overline{n}\}$, written as a word, with a bijection

$$\pi \colon E \cup \overline{E} \longrightarrow \{-(2n+1), -(2n-1), \ldots, -3, -1, 1, 3, \ldots, (2n-1), (2n+1)\}$$

[14] Alternatively, one may appeal to the unimodularity of the $2n$-dimensional braid arrangement fan in (Z, W)-coordinates discussed in Section 2.3.

that sends the letters of the word to $-(2n+1), \dots, -1, 1, \dots, (2n+1)$ in increasing order. For example, the permutation $12\overline{2}3\overline{1}\overline{3}0\overline{0}$ gives rise to the following bijection π:

$$12\overline{2}3\overline{1}\overline{3}0\overline{0} \longmapsto \pi = \begin{pmatrix} 1 & 2 & \overline{2} & 3 & \overline{1} & \overline{3} & 0 & \overline{0} \\ -7 & -5 & -3 & -1 & 1 & 3 & 5 & 7 \end{pmatrix}.$$

To the bijection π we associate a vector $u_\pi = (x, y) \in \mathbb{R}^E \times \mathbb{R}^E$ with coordinates $x_i = \pi(i)$ and $y_i = -\pi(\overline{i})$ for $i \in E$. Notice that u_π is on the hyperplane $\sum_{i \in E} x_i - \sum_{i \in E} y_i = 0$, so we may define $s_\pi = \sum_{i \in E} x_i = \sum_{i \in E} y_i$. Writing vectors $(x, y) \in \mathbb{R}^E \times \mathbb{R}^E$ in a $2 \times E$ table whose top and bottom rows are x and y respectively, we have, for example,

$$u_{12\overline{2}3\overline{1}\overline{3}0\overline{0}} = \boxed{\begin{matrix} 5 & -7 & -5 & -1 \\ -7 & -1 & 3 & -3 \end{matrix}}, \qquad s_\pi = -8.$$

Now, for each bipermutation $\mathcal{B}$ on E with only one occurrence of $k \in E$, let $\pi(\mathcal{B})$ be the permutation of $E \cup \overline{E}$ obtained by replacing the first and second occurrences of each $i \neq k$ with i and $\overline{i}$ respectively, and replacing k with $k\overline{k}$. Then define

$$v_{\mathcal{B}} = u_{\pi(\mathcal{B})} - s_{\pi(\mathcal{B})}(\mathbf{e}^k + \mathbf{f}^k),$$

where $\mathbf{e}^k$ and $\mathbf{f}^k$ are the kth unit vectors in the first and second copies of R^E. For example,

$$\begin{aligned} v_{1|2|3|1|3|0|0} &= u_{12\overline{2}3\overline{1}\overline{3}0\overline{0}} - s_{12\overline{2}3\overline{1}\overline{3}0\overline{0}}(\mathbf{e}^2 + \mathbf{f}^2) \\ &= \boxed{\begin{matrix} 5 & -7 & -5 & -1 \\ -7 & -1 & 3 & -3 \end{matrix}} + 8 \boxed{\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{matrix}} = \boxed{\begin{matrix} 5 & -7 & 3 & -1 \\ -7 & -1 & 11 & -3 \end{matrix}} \end{aligned}$$

The row sums of $v_{\mathcal{B}}$ equal 0, so $v_{\mathcal{B}} \in M_E \oplus M_E$ where M_E is the dual vector space to N_E.

Definition 2.19. The *bipermutohedron* of E is

$$\Pi_{E,E} := \mathrm{conv}\{v_{\mathcal{B}} \ : \ \mathcal{B} \text{ is a bipermutation on } E\} \subset M_E \oplus M_E.$$

Theorem 2.20. The bipermutohedral fan $\Sigma_{E,E}$ is the normal fan of the bipermutohedron $\Pi_{E,E}$.

Proof. Let $\mathcal{B}$ be a bipermutation on E. We claim that the cone of the normal fan of $\Pi_{E,E}$ corresponding to $v_{\mathcal{B}}$ is precisely the maximal cone $\sigma_{\mathcal{B}}$ of the bipermutohedral fan $\Sigma_{E,E}$:

$$\mathcal{N}_{\Pi_{E,E}}(v_{\mathcal{B}}) = \sigma_{\mathcal{B}}. \tag{2.7.1}$$

This will prove the desired result. It will also show that each $v_{\mathcal{B}}$ is indeed a vertex of the bipermutohedron $\Pi_{E,E}$.

$\subseteq$: Consider a linear functional $(z, w) \in \mathrm{N}_{E,E}$ in $\mathcal{N}_{\Pi_{E,E}}(v_{\mathcal{B}})$, so the (z, w)-minimal face of $\Pi_{E,E}$ contains the vertex $v_{\mathcal{B}}$. We need to show that $(z, w) \in \sigma_{\mathcal{B}}$.

For any adjacent letters i, j of $\mathcal{B}$, let $\mathcal{B}'$ be the bipermutation obtained by swapping them:

$$\mathcal{B} = \dots|i|j|\dots, \qquad \mathcal{B}' = \dots|j|i|\dots,$$

Notice that the tables of $v_{\mathcal{B}}$ and $v'_{\mathcal{B}}$ can only differ in columns i, j, and k, where k is the letter that is not repeated in $\mathcal{B}$. We use this fact to simplify the inequality $w(\mathcal{B}) \leqslant w(\mathcal{B}')$, rewriting it in terms of the coordinate system of chart C_k of $N_E \times N_E$:

$$Z_i = z_i - z_k, \qquad W_i = -(w_i - w_k) \qquad \text{for } i \in E.$$

There are eight cases:

Case 1: $i, j \in E - k$. The permutations $\pi = \pi(\mathcal{B})$ and $\pi' = \pi(\mathcal{B}')$ satisfy $\pi(i) = a-1, \pi(j) = a+1$ and $\pi'(i) = a+1$, $\pi'(j) = a-1$ for some a. Also $s(\pi') = s(\pi)$. Therefore

$$(z,w)\left(\begin{array}{ccccccc} & i & & k & & j & \\ \hline \cdot & a-1 & \cdot & -s & \cdot & a+1 & \cdot \\ \cdot & \cdot & \cdot & -s & \cdot & \cdot & \cdot \end{array}\right) \leqslant (z,w)\left(\begin{array}{ccccccc} & i & & k & & j & \\ \hline \cdot & a+1 & \cdot & -s & \cdot & a-1 & \cdot \\ \cdot & \cdot & \cdot & -s & \cdot & \cdot & \cdot \end{array}\right)$$

$$\begin{aligned} (a-1)z_i + (a+1)z_j &\leqslant (a+1)z_i + (a-1)z_j \\ z_j &\leqslant z_i \\ Z_j &\leqslant Z_i \end{aligned}$$

Case 2: $i, j \in \overline{E-k}$. This is the reverse of Case 1. Similarly, we have:

$$(z,w)\left(\begin{array}{ccccccc} & i & & k & & j & \\ \hline \cdot & \cdot & \cdot & -s & \cdot & \cdot & \cdot \\ \cdot & -(a-1) & \cdot & -s & \cdot & -(a+1) & \cdot \end{array}\right) \leqslant (z,w)\left(\begin{array}{ccccccc} & i & & k & & j & \\ \hline \cdot & \cdot & \cdot & -s & \cdot & \cdot & \cdot \\ \cdot & -(a+1) & \cdot & -s & \cdot & -(a-1) & \cdot \end{array}\right)$$

$$\begin{aligned} -(a-1)w_i - (a+1)w_j &\leqslant -(a+1)w_i - (a-1)w_j \\ -w_j &\leqslant -w_i \\ W_j &\leqslant W_i \end{aligned}$$

Case 3: $i \in E - k$ and $j \in \overline{E-k}$. Again, $\pi(i) = a-1, \pi(j) = a+1$ and $\pi'(i) = a+1$, $\pi'(j) = a-1$ for some a. Now we have that $s(\pi') = s(\pi) + 2$, so

$$(z,w)\left(\begin{array}{ccccccc} & i & & k & & j & \\ \hline \cdot & a-1 & \cdot & -s & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -s & \cdot & -(a+1) & \cdot \end{array}\right) \leqslant (z,w)\left(\begin{array}{ccccccc} & i & & k & & j & \\ \hline \cdot & a+1 & \cdot & -(s+2) & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -(s+2) & \cdot & -(a-1) & \cdot \end{array}\right)$$

$$\begin{aligned} (a-1)z_i - (a+1)w_j &\leqslant (a+1)z_i - (a-1)w_j - 2z_k - 2w_k \\ -(w_j - w_k) &\leqslant z_i - z_k \\ W_j &\leqslant Z_i \end{aligned}$$

Case 4: $i \in \overline{E-k}$ and $j \in E - k$. This is the reverse of Case 3. We obtain $Z_j \leqslant W_i$.

Case 5: $i = k$ and $j \in E - k$. Now π and π' satisfy $\pi(i) = a - 2$, $\pi(\bar{i}) = a$, $\pi(j) = a + 2$ and $\pi'(i) = a$, $\pi'(\bar{i}) = a + 2$, $\pi'(j) = a - 2$ for some a. In this case $s(\pi') = s(\pi) - 2$, so

$$
(z,w)\begin{array}{c} \begin{array}{ccccc} & i=k & & j & \end{array} \\ \left(\boxed{\begin{array}{ccccc} \cdot & (a-2)-s & \cdot & a+2 & \cdot \\ \cdot & -a-s & \cdot & \cdot & \cdot \end{array}}\right) \end{array} \leqslant (z,w)\begin{array}{c} \begin{array}{ccccc} & i & & j=k & \end{array} \\ \left(\boxed{\begin{array}{ccccc} \cdot & a-(s-2) & \cdot & a-2 & \cdot \\ \cdot & -(a+2)-(s-2) & \cdot & \cdot & \cdot \end{array}}\right) \end{array}
$$

$$
\begin{aligned}
(a-2-s)z_i - (a+s)w_i + (a+2)z_j &\leqslant (a+2-s)z_i - (a+s)w_i + (a-2)z_j \\
z_j &\leqslant z_i \\
Z_j &\leqslant Z_i
\end{aligned}
$$

Case 6: $i \in E - k$ and $j = k$. This is the reverse of Case 5. We obtain $Z_j \leqslant Z_i$.

Case 7: $i = k$ and $j \in \overline{E - k}$. An argument analogous to Case 5 gives $W_j \leqslant W_i$.

Case 8: $i \in \overline{E - k}$, $j = k$. This is the reverse of Case 7. We obtain $W_j \leqslant W_i$.

Applying the above analysis to each pair of adjacent letters of $\mathcal{B}$, we conclude that the relative order of $Z_0, \ldots, Z_n, W_0, \ldots, W_n$ is (weakly) the same as the opposite order of $0, \ldots, n, \overline{0}, \ldots, \overline{n}$ in $\pi(\mathcal{B})$. In particular, since i precedes $\bar{i}$ for all $i \in E$, we have that $Z_i \geqslant W_i$ for all i; that is, $\min_{i \in E}(z_i + w_i) = z_k + w_k$. We conclude that (z, w) is in the cone C_k and it satisfies the defining inequalities of $\sigma_{\mathcal{B}} \subset C_k$. Therefore $(z, w) \in \sigma_{\mathcal{B}}$, as desired.

$\supseteq$: Consider a point (z, w) in the interior of $\sigma_{\mathcal{B}}$. If (z, w) were not in the normal cone $\mathcal{N}_{\Pi_{E,E}}(v_{\mathcal{B}})$, then it would have to be in the normal cone $\mathcal{N}_{\Pi_{E,E}}(v_\sigma)$ for some other vertex $v_{\mathcal{B}'}$ corresponding to a bipermutation $\mathcal{B}' \neq \mathcal{B}$. But then $(z, w) \in \sigma_{\mathcal{B}'}$ by the first part of this proof, and this would mean that one maximal cone in the fan $\Sigma_{E,E}$ intersects the interior of another, a contradiction. We conclude that $\mathcal{N}_{\Pi_{E,E}}(v_{\mathcal{B}})$ contains $\operatorname{int} \sigma_{\mathcal{B}}$ and, being closed, it must contain all of $\sigma_{\mathcal{B}}$ as desired. $\square$

3. THE CONORMAL INTERSECTION THEORY OF A MATROID

In this section, we construct the *conormal fan* of a matroid M on E, and describe its Chow ring. Our running example will be the graphic matroid $\mathrm{M}(G)$ of the graph G of the square pyramid, whose dual is the graphic matroid of the dual graph $G^\perp$ shown in Figure 6.

3.1. **Homology and cohomology.** Throughout this section we fix a simplicial rational fan Σ in $\mathrm{N} = \mathbb{R} \otimes \mathrm{N}_{\mathbb{Z}}$. For each ray ρ in Σ, we write $\mathbf{e}_\rho$ for the primitive generator of ρ in $\mathrm{N}_{\mathbb{Z}}$, and introduce a variable x_ρ.

- Let $S(\Sigma)$ be the polynomial ring with real coefficients that has x_ρ as its variables, one for each ray ρ of Σ.
- Let $I(\Sigma)$ be the *Stanley-Reisner* ideal of $S(\Sigma)$, generated by the square-free monomials indexing the subsets of rays of Σ which do not generate a cone in Σ.

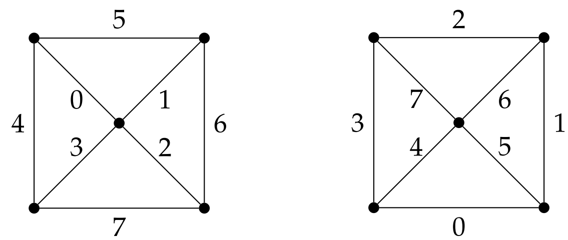


FIGURE 6. The graph G of the square pyramid and its dual graph $G^{\perp}$.

- Let $J(\Sigma)$ be the ideal of $S(\Sigma)$ generated by the linear forms $\sum_\rho \ell(\mathbf{e}_\rho)x_\rho$, where ℓ is any linear function on N and the sum is over all the rays in Σ.

Definition 3.1. The *Chow ring* of Σ, denoted $A(\Sigma)$, is the graded algebra $S(\Sigma)/(I(\Sigma)+J(\Sigma))$.

Billera [Bil89] constructed an isomorphism from the monomial quotient $S(\Sigma)/I(\Sigma)$ to the algebra of continuous piecewise polynomial functions on Σ by identifying the variable x_ρ with the piecewise linear *tent* or *Courant function* on Σ determined by the condition

$$x_\rho(\mathbf{e}_{\rho'}) = \begin{cases} 1, & \text{if } \rho \text{ is equal to } \rho', \\ 0, & \text{if } \rho \text{ is not equal to } \rho'. \end{cases}$$

Thus, under this isomorphism, a piecewise linear function ℓ on Σ is identified with the linear form

$$\ell = \sum_\rho \ell(\mathbf{e}_\rho)x_\rho.$$

We regard the elements of $A(\Sigma)$ as equivalence classes of piecewise polynomial functions on Σ, modulo the restrictions of global linear functions to Σ.

Brion [Bri96] showed that the Chow ring of the toric variety $X(\Sigma)$ of Σ with real coefficients is isomorphic to $A(\Sigma)$.[15] Under this isomorphism, the class of the torus orbit closure of a cone σ in Σ is identified with $\mathrm{mult}(\sigma)\, x_\sigma$, where x_σ is the monomial $\prod_{\rho\subseteq\sigma} x_\rho$ and $\mathrm{mult}(\sigma)$ is the index of the subgroup

$$\Big(\sum_{\rho\subseteq\sigma} \mathbb{Z}\,\mathbf{e}_\rho\Big) \subseteq \mathrm{N}_\mathbb{Z} \cap \Big(\sum_{\rho\subseteq\sigma} \mathbb{R}\,\mathbf{e}_\rho\Big).$$

All the fans appearing in this section will be unimodular, so $\mathrm{mult}(\sigma)=1$ for every σ in Σ.

We write $\Sigma(k)$ for the set of k-dimensional cones in Σ. A *k-dimensional Minkowski weight* on Σ is a real-valued function ω on $\Sigma(k)$ that satisfies the *balancing condition*: For every $(k-1)$-dimensional cone τ in Σ,

$$\sum_{\tau\subset\sigma} \omega(\sigma)\mathbf{e}_{\sigma/\tau} = 0 \text{ in the quotient space } \mathrm{N}/\operatorname{span}(\tau),$$

[15] When Σ is complete, this description of the Chow ring can be deduced from a classical result of Danilov and Jurkiewicz [CLS11, Theorem 12.5.3].

where $\mathbf{e}_{\sigma/\tau}$ is the primitive generator of the ray $(\sigma + \operatorname{span}(\tau))/\operatorname{span}(\tau)$. We say that w is *positive* if $w(\sigma)$ is positive for every σ in $\Sigma(k)$. We write $\mathrm{MW}_k(\Sigma)$ for the space of k-dimensional Minkowski weights on Σ, and set

$$\mathrm{MW}(\Sigma) = \bigoplus_{k \geqslant 0} \mathrm{MW}_k(\Sigma).$$

We will make use of the basic fact that the Chow group of a toric variety is generated by the classes of torus orbit closures [CLS11, Lemma 12.5.1]. Thus, there is an injective linear map from the dual of $A^k(\Sigma)$ to the space of k-dimensional weights on Σ, whose image turns out to be $\mathrm{MW}_k(\Sigma)$, as noted in [FS97]. Explicitly, the inverse isomorphism from the image is

$$\mathrm{MW}_k(\Sigma) \longrightarrow \operatorname{Hom}(A^k(\Sigma), \mathbb{R}), \qquad w \longmapsto (\operatorname{mult}(\sigma) x_\sigma \longmapsto w(\sigma)).$$

Following [AHK18, Section 5], we define the *cap product*, denoted $\eta \cap w$, using the composition

$$A^\ell(\Sigma) \longrightarrow \operatorname{Hom}(A^{k-\ell}(\Sigma), A^k(\Sigma)) \longrightarrow \operatorname{Hom}(\mathrm{MW}_k(\Sigma), \mathrm{MW}_{k-\ell}(\Sigma)), \quad \eta \longmapsto (w \longmapsto \eta \cap w),$$

where the first map is given by the multiplication in the Chow ring of Σ. In short, $\mathrm{MW}(\Sigma)$ has the structure of a graded $A(\Sigma)$-module given by the isomorphism $\mathrm{MW}(\Sigma) \simeq \operatorname{Hom}(A(\Sigma), \mathbb{R})$.

Let $f : \Sigma \to \Sigma'$ be a morphism of simplicial fans. The pullback of functions define the *pullback homomorphism* between the Chow rings

$$f^* : A(\Sigma') \longrightarrow A(\Sigma),$$

whose dual is the *pushforward homomorphism* between the space of Minkowski weights

$$f_* : \mathrm{MW}(\Sigma) \longrightarrow \mathrm{MW}(\Sigma').$$

Since f^* is a homomorphism of graded rings, f_* is a homomorphism of graded modules. In other words, the pullback and the pushforward homomorphisms satisfy the *projection formula*

$$\eta \cap f_* w = f_*(f^* \eta \cap w).$$

3.2. **The Bergman fan of a matroid.** The *Bergman fan* of a matroid M on E, denoted Σ_{M}, is the r-dimensional subfan of the n-dimensional permutohedral fan Σ_E whose underlying set is the *tropical linear space*

$$\operatorname{trop}(\mathrm{M}) = \Big\{ z \,|\, \min_{i \in C}(z_i) \text{ is achieved at least twice for every circuit } C \text{ of } \mathrm{M} \Big\} \subseteq \mathrm{N}_E\,.$$

The Bergman fan of M is equipped with the piecewise linear functions

$$\alpha_j = \max_{i \in E}(z_j - z_i),$$

and the space of linear functions on the Bergman fan is spanned by the differences

$$\alpha_i - \alpha_j = z_i - z_j.$$

Note that $\mathrm{trop}(\mathrm{M})$ is nonempty if and only if M is *loopless*. In the remainder of this section, we suppose that M has no loops. In this case, the Bergman fan of M is the induced subfan of Σ_E generated by the rays corresponding to the nonempty proper flats of M [AK06].

Proposition 3.2. The Bergman fan of M is the unimodular fan in N_E with the cones

$$\sigma_{\mathcal{F}} = \mathrm{cone}\{\mathbf{e}_F\}_{F \in \mathcal{F}}, \ \ \text{where } \mathcal{F} \text{ is a flag of flats of } \mathrm{M}.$$

The most important geometric property of Σ_{M} is the following description of its top-dimensional Minkowski weights. For a proof, see, for example, [AHK18, Proposition 5.2].

Proposition 3.3. An r-dimensional weight on Σ_{M} is balanced if and only if it is constant.

We write 1_{M} for the *fundamental weight* on Σ_{M}, the r-dimensional Minkowski weight on the Bergman fan that has the constant value 1.

3.3. **The Chow ring of the Bergman fan.** In the context of matroids, for simplicity, we set

$$S_{\mathrm{M}} = S(\Sigma_{\mathrm{M}}), \quad I_{\mathrm{M}} = I(\Sigma_{\mathrm{M}}), \quad J_{\mathrm{M}} = J(\Sigma_{\mathrm{M}}), \quad A_{\mathrm{M}} = A(\Sigma_{\mathrm{M}}).$$

We identify the elements of $S_{\mathrm{M}}/I_{\mathrm{M}}$ with the piecewise linear functions on Σ_{M} as before.

Let x_F be the variable of the polynomial ring corresponding to the ray generated by $\mathbf{e}_F$ in the Bergman fan. For any set $\mathcal{F}$ of nonempty proper flats of M, we write $x_{\mathcal{F}}$ for the monomial

$$x_{\mathcal{F}} = \prod_{F \in \mathcal{F}} x_F.$$

The variable x_F, viewed as a piecewise linear function on the Bergman fan, is given by

$$x_F(\mathbf{e}_{F'}) = \begin{cases} 1, & \text{if } F \text{ is equal to } F', \\ 0, & \text{if } F \text{ is not equal to } F', \end{cases}$$

and hence the piecewise linear function α_j on the Bergman fan satisfies the identity

$$\alpha_j = \sum_F \alpha_j(\mathbf{e}_F) x_F = \sum_{j \in F} x_F.$$

Thus, in the above notation,

- S_{M} is the ring of polynomials in the variables x_F, where F is a nonempty proper flat of M,
- I_{M} is the ideal generated by the monomials $x_{\mathcal{F}}$, where $\mathcal{F}$ is not a flag, and
- J_{M} is the ideal generated by the linear forms $\alpha_i - \alpha_j$, for any i and j in E.

We write α for the common equivalence class of α_j in the Chow ring of the Bergman fan.

Definition 3.4. The fundamental weight 1_{M} defines the *degree map*

$$\deg : A^r_{\mathrm{M}} \longrightarrow \mathbb{R}, \quad x_{\mathcal{F}} \longmapsto x_{\mathcal{F}} \cap 1_{\mathrm{M}} = \begin{cases} 1 & \text{if } \mathcal{F} \text{ is a flag}, \\ 0 & \text{if } \mathcal{F} \text{ is not a flag}. \end{cases}$$

By Proposition 3.3, the degree map is an isomorphism. In other words, for any maximal flag $\mathcal{F}$ of nonempty proper flats of M, the class of the monomial $x_{\mathcal{F}}$ in the Chow ring of the Bergman fan of M is nonzero and does not depend on $\mathcal{F}$.

3.4. **The conormal fan of a matroid.** The *conormal fan* of a matroid M on E, denoted $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$, is the $(n-1)$-dimensional subfan of the $2n$-dimensional bipermutohedral fan $\Sigma_{E,E}$ whose support is the product of tropical linear spaces

$$|\Sigma_{\mathrm{M},\mathrm{M}^\perp}| = \mathrm{trop}(\mathrm{M}) \times \mathrm{trop}(\mathrm{M}^\perp).$$

Equivalently, the conormal fan is the largest subfan of the bipermutohedral fan for which the projections to the factors are morphisms of fans

$$\pi : \Sigma_{\mathrm{M},\mathrm{M}^\perp} \longrightarrow \Sigma_{\mathrm{M}} \quad \text{and} \quad \overline{\pi} : \Sigma_{\mathrm{M},\mathrm{M}^\perp} \longrightarrow \Sigma_{\mathrm{M}^\perp}.$$

The addition map $(z,w) \mapsto z+w$ is also a morphism of fans $\Sigma_{\mathrm{M},\mathrm{M}^\perp} \to \Gamma_E$.

The conormal fan of M is equipped with the piecewise linear functions

$$\gamma_j = \max_{i\in E}(z_j - z_i), \qquad \overline{\gamma}_j = \max_{i\in E}(w_j - w_i), \qquad \delta_j = \max_{i\in E}(z_j + w_j - z_i - w_i),$$

which are the pullbacks of α_j under the projections π and π' and the addition map, respectively. The space of linear functions on the conormal fan is spanned by the differences

$$\gamma_i - \gamma_j = z_i - z_j \quad \text{and} \quad \overline{\gamma}_i - \overline{\gamma}_j = w_i - w_j.$$

Note that the support of the conormal fan of M is nonempty if and only if M is *loopless* and *coloopless*. In the remainder of this section, we suppose that M has no loops and no coloops.

Definition 3.5. A *biflat* $F|G$ of M consists of a flat F of M and a flat G of $\mathrm{M}^\perp$ that form a bisubset; that is, they are nonempty, they are not both equal to E, and their union is E. A *biflag* of M is a flag of biflats.

We give an analog of Proposition 3.2 for conormal fans in terms of biflats.

Proposition 3.6. The conormal fan of M is the unimodular fan in $\mathrm{N}_{E,E}$ with the cones

$$\sigma_{\mathcal{F}|\mathcal{G}} = \mathrm{cone}\{\mathbf{e}_{F|G}\}_{F|G \in \mathcal{F}|\mathcal{G}}, \ \text{ for } \mathcal{F}|\mathcal{G} \text{ a flag of biflats of } \mathrm{M}.$$

Proof. The proof is straightforward, given Corollary 2.17 and Proposition 3.2: If $\mathcal{F}|\mathcal{G}$ is a flag of biflats of M, then $\mathcal{F}$ is an increasing sequence of flats of M and $\mathcal{G}$ is a decreasing sequence of flats of $\mathrm{M}^\perp$, and hence

$$\sigma_{\mathcal{F}|\mathcal{G}} \subseteq \sigma_{\mathcal{F}} \times \sigma_{\mathcal{G}} \in \Sigma_{\mathrm{M}} \times \Sigma_{\mathrm{M}^\perp}.$$

Therefore, the conormal fan of M contains the induced subfan of $\Sigma_{E,E}$ generated by the rays corresponding to the biflats of M. The other inclusion follows from the easy implication

$$\mathbf{e}_{F|G} \text{ is in the support of the conormal fan of } \mathrm{M} \Longrightarrow F|G \text{ is a biflat of } \mathrm{M}. \qquad \square$$

We also have the following analog of Proposition 3.3 for conormal fans.

Proposition 3.7. An $(n-1)$-dimensional weight on $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ is balanced if and only if it is constant.

We write $1_{\mathrm{M},\mathrm{M}^\perp}$ for the *fundamental weight* on $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$, the top-dimensional Minkowski weight on the conormal fan that has the constant value 1.

Proof. Proposition 3.3 applied to M and $\mathrm{M}^\perp$ shows that a top-dimensional weight on $\Sigma_\mathrm{M} \times \Sigma_{\mathrm{M}^\perp}$ satisfies the balancing condition if and only if it is constant. This property of the fan remains invariant under any subdivision of its support, as shown in [GKM09, Section 2]. $\square$

For our purposes, the product of the Bergman fans of M and $\mathrm{M}^\perp$ has a shortcoming: The addition map need not be a morphism from the product to the fan Γ_E. Thus, in general, we cannot define the class of δ_j in the Chow ring of the product. This is our motivation for subdividing it further, to obtain the conormal fan $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$.

Example 3.8. Let M and $\mathrm{M}^\perp$ be the graphic matroids of the graphs in Figure 6. Consider the cone $\sigma_\mathcal{F} \times \sigma_\mathcal{G}$ in the product of Bergman fans of M and $\mathrm{M}^\perp$, where

$$\mathcal{F} = (\varnothing \subsetneq 1 \subsetneq 015 \subsetneq 01345 \subsetneq E) \quad \text{and} \quad \mathcal{G} = (\varnothing \subsetneq 2 \subsetneq 267 \subsetneq 12567 \subsetneq E).$$

This cone is subdivided into the chambers of $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ corresponding to the biflags

$$\begin{array}{|cc|ccccccccccc|cc|}
\hline
\varnothing & \subsetneq & 1 & \subseteq & 015 & \subseteq & 01345 & \subseteq & 01345 & \subseteq & 01345 & \subseteq & E & \subseteq & E \\
E & \supseteq & E & \supseteq & E & \supseteq & E & \supseteq & 12567 & \supseteq & 267 & \supseteq & 2 & \supsetneq & \varnothing \\
\hline
\end{array},$$

$$\begin{array}{|cc|ccccccccccc|cc|}
\hline
\varnothing & \subsetneq & 1 & \subseteq & 015 & \subseteq & 01345 & \subseteq & 01345 & \subseteq & E & \subseteq & E & \subseteq & E \\
E & \supseteq & E & \supseteq & E & \supseteq & 12567 & \supseteq & 267 & \supseteq & 267 & \supseteq & 2 & \supsetneq & \varnothing \\
\hline
\end{array},$$

$$\begin{array}{|cc|ccccccccccc|cc|}
\hline
\varnothing & \subsetneq & 1 & \subseteq & 015 & \subseteq & 01345 & \subseteq & E & \subseteq & E & \subseteq & E & \subseteq & E \\
E & \supseteq & E & \supseteq & E & \supseteq & 12567 & \supseteq & 12567 & \supseteq & 267 & \supseteq & 2 & \supsetneq & \varnothing \\
\hline
\end{array}.$$

If (z,w) is inside the first chamber, then the minimum of $z_i + w_i$ is attained by $z_6 + w_6 = z_7 + w_7$, and hence $z + w$ is in the cone σ_{012345}. If (z,w) is inside the second or the third chamber, then the minimum of $z_i + w_i$ is attained by $z_3 + w_3 = z_4 + w_4$, and hence $z + w$ is in the cone σ_{012567}. Thus, the product cone does not map into a cone in Γ_E under the addition map.

Recall from Definition 2.5 that the cotangent fan Ω_E is the subfan of $\Sigma_{E,E}$ with support

$$\mathrm{trop}(\delta) = \Big\{(z,w) \mid \min_{i \in E}(z_i + w_i) \text{ is achieved at least twice}\Big\} \subseteq \mathrm{N}_{E,E}\,.$$

In other words, the cotangent fan is the collection of cones $\sigma_\mathcal{B}$ for bisequences $\mathcal{B}$ on E, where at least two elements of E appear exactly once in $\mathcal{B}$. We show that the cotangent fan contains all the conormal fans of matroids on E.

Proposition 3.9. For any matroid M on E, we have $\mathrm{trop}(\mathrm{M}) \times \mathrm{trop}(\mathrm{M}^\perp) \subseteq \mathrm{trop}(\delta)$.

In other words, if the minimum of $(z_i)_{i \in C}$ is achieved at least twice for every circuit C of M and the minimum of $(w_i)_{i \in C^\perp}$ is achieved at least twice for every circuit $C^\perp$ of $\mathrm{M}^\perp$, then the minimum of $(z_i + w_i)_{i \in E}$ is achieved at least twice. We deduce Proposition 3.9 from Proposition 3.14 below, a stronger statement on the flags of biflats of M. The notion of gaps introduced here for Proposition 3.14 will be useful in Section 4.

Let $\mathcal{F}|\mathcal{G}$ be a flag of biflats of M. As before, we write $\mathcal{F}$ and $\mathcal{G}$ for the sequences

$$\begin{aligned} \mathcal{F} &= (\varnothing \subsetneq F_1 \subseteq \cdots \subseteq F_k \subseteq E), \text{ where } F_j \text{ are the first parts of the biflats in } \mathcal{F}|\mathcal{G}, \\ \mathcal{G} &= (E \supseteq G_1 \supseteq \cdots \supseteq G_k \supsetneq \varnothing), \text{ where } G_j \text{ are the second parts of the biflats in } \mathcal{F}|\mathcal{G}, \end{aligned}$$

where k is the length of $\mathcal{F}|\mathcal{G}$. Thus, the bisequence $\mathcal{B}(\mathcal{F}|\mathcal{G})$ from Proposition 2.16 can be written

$$B_0|B_1|\cdots|B_k, \text{ where } B_j = (F_{j+1} - F_j) \cup (G_j - G_{j+1}).$$

Definition 3.10. The *gap sequence* of $\mathcal{F}|\mathcal{G}$, denoted $\mathcal{D}(\mathcal{F}|\mathcal{G})$, is the sequence of *gaps*

$$D_0|D_1|\cdots|D_k, \text{ where } D_j = (F_{j+1} - F_j) \cap (G_j - G_{j+1}).$$

Note that D_j consists of the elements of B_j that appear exactly once in the bisequence $\mathcal{B}(\mathcal{F}|\mathcal{G})$.

Example 3.11. The three maximal flags of biflats shown in Example 3.8 have the gap sequences

$$\varnothing|\varnothing|\varnothing|\varnothing|\varnothing|67|\varnothing, \quad \varnothing|\varnothing|34|\varnothing|\varnothing|\varnothing|\varnothing, \quad \varnothing|\varnothing|34|\varnothing|\varnothing|\varnothing|\varnothing.$$

We show in Proposition 3.17 that any maximal flag of biflats has a unique nonempty gap.

Lemma 3.12. The complement of the gap D_j in E is the union of F_j and G_{j+1}.

Therefore, by Proposition 2.15, at least one of the gaps of $\mathcal{F}|\mathcal{G}$ must be nonempty.

Proof. Since $F_j|G_j$ and $F_{j+1}|G_{j+1}$ are bisubsets, we have $G_j^c \subseteq F_j$ and $F_{j+1}^c \subseteq G_{j+1}$. Thus,

$$D_j^c = (F_{j+1} \cap F_j^c \cap G_j \cap G_{j+1}^c)^c = F_{j+1}^c \cup F_j \cup G_j^c \cup G_{j+1} = F_j \cup G_{j+1}. \qquad \square$$

Lemma 3.13. Let $e \in E$. There exists an index i for which $e \in F_i \cap G_i$ if and only if e is not in any gap. In symbols, the union of the gaps of $\mathcal{F}|\mathcal{G}$ is

$$\bigsqcup_{j=0}^{k} D_j = E - \bigcup_{i=1}^{k} (F_i \cap G_i)\,.$$

Proof. First suppose $e \in F_i \cap G_i$. Then $e \in F_j$ for all $j \geqslant i$, which means $e \notin D_j$ for $i \leqslant j \leqslant k$. Dually, $e \in G_j$ for all $j \leqslant i$, so $e \notin D_j$ for all $0 \leqslant i \leqslant j-1$.

Now suppose e is not in any gap, and consider the index $1 \leqslant i \leqslant k+1$ for which $e \in F_i - F_{i-1}$. Since $e \in F_{i-1} \cup G_i$, we must have $e \in G_i$ and hence $e \in F_i \cap G_i$. $\square$

Proposition 3.14. Every nonempty gap of a biflag $\mathcal{F}|\mathcal{G}$ of M has at least two distinct elements.

Proof. Recall that, for any matroid, the complement of any hyperplane is a cocircuit [Oxl11, Proposition 2.1.6] and that any flat is an intersection of hyperplanes [Oxl11, Proposition 1.7.8].

Since the complement of a gap of $\mathcal{F}|\mathcal{G}$ is the union of a flat and a coflat by Lemma 3.12, we may write the gap as the intersection

$$\Big(\bigcup_{C\in\mathcal{C}} C\Big) \cap \Big(\bigcup_{C^\perp\in\mathcal{C}^\perp} C^\perp\Big),$$

where $\mathcal{C}$ is a collection of circuits and $\mathcal{C}^\perp$ is a collection of cocircuits. Thus, if the gap is nonempty, there are $C \in \mathcal{C}$ and $C^\perp \in \mathcal{C}^\perp$ that intersect nontrivially. Now the first statement follows from the classical fact that the intersection of a circuit and a cocircuit is either empty or contains at least two elements [Oxl11, Proposition 2.11]. □

For any biflag $\mathcal{F}|\mathcal{G}$, there are at least two elements of E that appear exactly once in the bisequence $\mathcal{B}(\mathcal{F}|\mathcal{G})$; therefore

$$\mathrm{trop}(\mathrm{M}) \times \mathrm{trop}(\mathrm{M}^\perp) \subseteq \mathrm{trop}(\delta),$$

proving Proposition 3.9.

We will often use the following restatement of Proposition 3.14. Recall that $|E| = n+1$.

Lemma 3.15. The union of a flat and a coflat cannot have exactly n elements.

For later use, we record here another elementary property of the flags of biflats of a matroid.

Definition 3.16. The *jump sets* of $\mathcal{F}$ and $\mathcal{G}$ are the sets of indices

$$\mathsf{J}(\mathcal{F}) = \{j \,|\, 0 \leqslant j \leqslant k \text{ and } F_j \neq F_{j+1}\} \text{ and } \mathsf{J}(\mathcal{G}) = \{j \,|\, 0 \leqslant j \leqslant k \text{ and } G_j \neq G_{j+1}\}.$$

The elements of $\mathsf{J}(\mathcal{F}) \cap \mathsf{J}(\mathcal{G})$ are called the *double jumps* of $\mathcal{F}|\mathcal{G}$.

The double jumps are colored blue in the table of $\mathcal{F}|\mathcal{G}$, as shown in Example 3.8. Clearly, j is a double jump whenever the corresponding gap D_j is nonempty. We show that the converse holds when $\mathcal{F}|\mathcal{G}$ is maximal.

Proposition 3.17. Every maximal flag of biflats $\mathcal{F}|\mathcal{G}$ of M has a unique double jump. Ignoring repetitions, $\mathcal{F}$ and $\mathcal{G}$ are complete flags of non-zero flats in M and $\mathrm{M}^\perp$, respectively.

In particular, every maximal flag of biflats $\mathcal{F}|\mathcal{G}$ of M has a unique nonempty gap.

Proof. Recall that at least one of the gaps of $\mathcal{F}|\mathcal{G}$ is nonempty. In addition, since tropical linear spaces are pure-dimensional, the length of any maximal flag of biflats must be $n-1$. Thus,

$$|\mathsf{J}(\mathcal{F}) \cap \mathsf{J}(\mathcal{G})| \geqslant 1 \text{ and } |\mathsf{J}(\mathcal{F}) \cup \mathsf{J}(\mathcal{G})| = n.$$

On the other hand, writing $r+1$ for the rank of M as before, we have

$$|\mathsf{J}(\mathcal{F})| \leqslant r+1 \text{ and } |\mathsf{J}(\mathcal{G})| \leqslant n-r.$$

Therefore, $n+1 \leqslant |\mathsf{J}(\mathcal{F}) \cup \mathsf{J}(\mathcal{G})| + |\mathsf{J}(\mathcal{F}) \cap \mathsf{J}(\mathcal{G})| = |\mathsf{J}(\mathcal{F})| + |\mathsf{J}(\mathcal{G})| \leqslant n+1$, and hence

$$|\mathsf{J}(\mathcal{F})| = r+1, \quad |\mathsf{J}(\mathcal{G})| = n-r \text{ and } |\mathsf{J}(\mathcal{F}) \cap \mathsf{J}(\mathcal{G})| = 1$$

which imply the desired results. □

3.5. **The Chow ring of the conormal fan.** For notational simplicity, we set

$$S_{\mathrm{M},\mathrm{M}^\perp} = S(\Sigma_{\mathrm{M},\mathrm{M}^\perp}), \quad I_{\mathrm{M},\mathrm{M}^\perp} = I(\Sigma_{\mathrm{M},\mathrm{M}^\perp}), \quad J_{\mathrm{M},\mathrm{M}^\perp} = J(\Sigma_{\mathrm{M},\mathrm{M}^\perp}), \quad A_{\mathrm{M},\mathrm{M}^\perp} = A(\Sigma_{\mathrm{M},\mathrm{M}^\perp}).$$

We identify the elements of $S_{\mathrm{M},\mathrm{M}^\perp}/I_{\mathrm{M},\mathrm{M}^\perp}$ with the piecewise linear functions on the conormal fan.

Let $x_{F|G}$ be the variable of the polynomial ring corresponding to the ray generated by $\mathbf{e}_{F|G}$ in the conormal fan. For any set $\mathcal{F}|\mathcal{G}$ of biflats of M, we write $x_{\mathcal{F}|\mathcal{G}}$ for the monomial

$$x_{\mathcal{F}|\mathcal{G}} = \prod_{F|G \in \mathcal{F}|\mathcal{G}} x_{F|G}.$$

We note that the piecewise linear function δ_j on the conormal fan satisfies the identity

$$\delta_j = \sum_{F|G} \delta_j(\mathbf{e}_{F|G}) x_{F|G} = \sum_{j \in F \cap G} x_{F|G}.$$

Similarly, the piecewise linear functions γ_j and $\overline{\gamma}_j$ satisfy the identities

$$\gamma_j = \sum_{j \in F \neq E} x_{F|G} \ \text{ and } \ \overline{\gamma}_j = \sum_{j \in G \neq E} x_{F|G}.$$

Thus, in the above notation,

- $S_{\mathrm{M},\mathrm{M}^\perp}$ is the ring of polynomials in the variables $x_{F|G}$, where $F|G$ is a biflat of M,
- $I_{\mathrm{M},\mathrm{M}^\perp}$ is the ideal generated by the monomials $x_{\mathcal{F}|\mathcal{G}}$, where $\mathcal{F}|\mathcal{G}$ is not a biflag, and
- $J_{\mathrm{M},\mathrm{M}^\perp}$ is the ideal generated by the linear forms $\gamma_i - \gamma_j$ and $\overline{\gamma}_i - \overline{\gamma}_j$, for any i and j in E.

We write γ, $\overline{\gamma}$, and δ, respectively, for the equivalence classes of γ_j, $\overline{\gamma}_j$, and δ_j in the Chow ring of the conormal fan.

Definition 3.18. The fundamental weight $1_{\mathrm{M},\mathrm{M}^\perp}$ of the conormal fan defines the *degree map*

$$\deg : A^{n-1}_{\mathrm{M},\mathrm{M}^\perp} \longrightarrow \mathbb{R}, \quad x_{\mathcal{F}|\mathcal{G}} \longmapsto x_{\mathcal{F}|\mathcal{G}} \cap 1_{\mathrm{M},\mathrm{M}^\perp} = \begin{cases} 1 & \text{if } \mathcal{F}|\mathcal{G} \text{ is a biflag,} \\ 0 & \text{if } \mathcal{F}|\mathcal{G} \text{ is not a biflag.} \end{cases}$$

By Proposition 3.7, the degree map is a linear isomorphism. In other words, for maximal flag of biflats $\mathcal{F}|\mathcal{G}$ of M, the class of the monomial $x_{\mathcal{F}|\mathcal{G}}$ in the Chow ring of the conormal fan of M is nonzero and does not depend on $\mathcal{F}|\mathcal{G}$.

Recall that the projection π is a morphism from the conormal fan of M to the Bergman fan of M. The projection has the special property that the image of a cone in the conormal fan is a cone in the Bergman fan (and not just contained in one). This property leads to the following simple description of the pullback $\pi^* : A_{\mathrm{M}} \to A_{\mathrm{M},\mathrm{M}^\perp}$.

Proposition 3.19. For any flag of nonempty proper flats $\mathcal{F}$ of M,

$$\pi^*(x_{\mathcal{F}}) = \sum_{\mathcal{G}} x_{\mathcal{F}|\mathcal{G}},$$

where the sum is over all decreasing sequences $\mathcal{G}$ such that $\mathcal{F}|\mathcal{G}$ is a flag of biflats of M.

Dually, the pushforward of any Minkowski weight w on the conormal fan is given by

$$\pi_*(w)(\sigma_{\mathcal{F}}) = \sum_{\mathcal{G}} w(\sigma_{\mathcal{F}|\mathcal{G}}),$$

where the sum is over all decreasing sequences $\mathcal{G}$ such that $\mathcal{F}|\mathcal{G}$ is a flag of biflats of M.

Proof. Since $\pi(\mathbf{e}_{F|G}) = \mathbf{e}_F$, the pullback of the piecewise linear function x_F satisfies

$$\pi^*(x_F) = \sum_G x_{F|G},$$

where the sum is over all G such that $F|G$ is a biflat of M. Thus, for any given $\mathcal{F}$,

$$\pi^*(x_{\mathcal{F}}) = \prod_{F\in\mathcal{F}} \pi^*(x_F) = \sum_{\mathcal{G}} x_{\mathcal{F}|\mathcal{G}},$$

where the sum is over all decreasing sequences $\mathcal{G}$ such that $\mathcal{F}|\mathcal{G}$ is a flag of biflats of M. □

4. Degree computations in the Chow ring of the conormal fan

Recall that the *beta invariant* of a matroid M of rank $r+1$ is

$$\beta(\mathrm{M}) \coloneqq (-1)^r\, \overline{\chi}_{\mathrm{M}}(1).$$

Given a strictly increasing flag of flats $\mathcal{F} = \{\varnothing \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E\}$, the *beta invariant of $\mathcal{F}$ in* M is

$$\beta(\mathrm{M}[\mathcal{F}]) \coloneqq \prod_{i=1}^{k+1} \beta(\mathrm{M}[F_{i-1}, F_i]), \tag{4.0.1}$$

where $\beta(\mathrm{M}[F_{i-1}, F_i])$ is the beta invariant of the matroid minor $\mathrm{M}(i) = \mathrm{M}[F_{i-1}, F_i] = M|F_i/F_{i-1}$ for $1 \leqslant i \leqslant k+1$.

The goal of this section is to prove Propositions 4.8 and 4.18, which state that

$$\deg(\delta^{n-1}) = \beta(\mathrm{M})$$

and, more generally, that for any strictly increasing flag of flats $\mathcal{F}$ in M of length k,

$$\deg(\pi^*(x_{\mathcal{F}})\delta^{n-k-1}) = \sum_{\mathcal{F}|\mathcal{G} \text{ biflag}} \deg(x_{\mathcal{F}|\mathcal{G}}\delta^{n-k-1}) = \beta(\mathrm{M}[\mathcal{F}]),$$

where $\pi^* : A_{\mathrm{M}} \to A_{\mathrm{M},\mathrm{M}^\perp}$ is the pullback of the projection map $\pi : \Sigma_{\mathrm{M},\mathrm{M}^\perp} \to \Sigma_{\mathrm{M}}$. Thus we seek to compute $x_{\mathcal{F}|\mathcal{G}}\, \delta^{n-k-1}$ in the Chow ring of $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$. This will require us to study more closely the combinatorial structure of conormal fans, and develop algebraic combinatorial techniques for computing in their Chow rings. We do so in this section.

4.1. **Canonical expansions in the Chow ring of the conormal fan.** In order to compute the degree of δ^{n-1} in the Chow ring $A_{\mathrm{M},\mathrm{M}^\perp}$ – or more generally the degree of $x_{\mathcal{F}|\mathcal{G}}\, \delta^{n-k-1}$ for a k-biflag $\mathcal{F}|\mathcal{G}$ — we seek to express it as a sum of square-free monomials, each of which have degree one by Definition 3.4. One fundamental feature of this computation, which is simultaneously an advantage and a difficulty, is that there are many different ways to carry it out, since we may choose from $n+1$ different expressions for δ; namely $\delta = \delta_i$ for all $i \in E$. To have control over the computation, we require some structure amidst that freedom. Thus we prescribe a canonical way of expressing δ^m (and more generally, $x_{\mathcal{F}|\mathcal{G}}\,\delta^m$) for each m.

Definition 4.1. (*Canonical expansion of $x_{\mathcal{F}|\mathcal{G}}\,\delta^m$.*) For a nonzero monomial $x_{\mathcal{F}|\mathcal{G}}$ in $A_{\mathrm{M},\mathrm{M}^\perp}$, let

$$e = e(\mathcal{F}|\mathcal{G}) := \max\big(E - \bigcup_{i=1}^{k}(F_i \cap G_i)\big) = \max\big(\bigsqcup_{j=0}^{k} D_j\big)$$

be the largest gap element of $\mathcal{F}|\mathcal{G}$, which exists thanks to Lemma 3.13. Define the *canonical expansion* of $x_{\mathcal{F}|\mathcal{G}}\,\delta$ to be

$$x_{\mathcal{F}|\mathcal{G}}\,\delta = x_{\mathcal{F}|\mathcal{G}}\,\delta_e = \sum_{\substack{F|G \in \mathsf{R}_{\mathrm{M},\mathrm{M}^\perp} \\ e \in F \cap G}} x_{\mathcal{F}|\mathcal{G}} x_{F|G}. \tag{4.1.1}$$

This is a sum of monomials in $A_{\mathrm{M},\mathrm{M}^\perp}$. Thus we may recursively obtain the canonical expansion of $x_{\mathcal{F}|\mathcal{G}}\,\delta^m$ for $m \geqslant 1$ by multiplying each monomial in the canonical expansion of $x_{\mathcal{F}|\mathcal{G}}\,\delta^{m-1}$ by δ, again using the canonical expansion.

Note that some or all of the summands in (4.1.1) may equal 0 in the Chow ring $A_{\mathrm{M},\mathrm{M}^\perp}$. The following lemma describes the non-zero terms.

Lemma 4.2. The canonical expansion of $x_{\mathcal{F}|\mathcal{G}}\,\delta$ is the sum of the monomials $x_{\mathcal{F}\cup F|\mathcal{G}\cup G}$ corresponding to the cones of the form $\sigma_{\mathcal{F}\cup F|\mathcal{G}\cup G} \supsetneq \sigma_{\mathcal{F}|\mathcal{G}}$ such that $e = e(\mathcal{F}|\mathcal{G}) \in F \cap G$. If e is in gap D_j, we must have $F_j \subseteq F \subseteq F_{j+1}$, $G_j \supseteq G \supseteq G_{j+1}$.

Proof. The first statement follows directly from the definitions. If $\sigma_{\mathcal{F}\cup F|\mathcal{G}\cup G}$ is a cone with $e \in F \cap G$, then $e \notin F_j$ and $e \notin G_{j+1}$ imply that the pair $F|G$ must be added in between indices j and $j+1$ of $\sigma_{\mathcal{F}|\mathcal{G}}$. Conversely, any such pair arises in this expansion. □

We may think of the canonical expansion of δ^m as a recursive procedure to produce a list of m-dimensional cones in the conormal fan $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$, where each cone is built up one ray at a time according to the rules prescribed in Lemma 4.2.

Example 4.3. For the graph G of the square pyramid in Figure 6, the canonical expansion of the highest non-zero power of δ in $A_{\mathrm{M},\mathrm{M}^\perp}$, namely $\delta^{n-1} = \delta^6$, is

$$\begin{aligned} \delta^6 \quad = \quad & x_{6|E}\, x_{56|E}\, x_{4567|E}\, x_{E|23467}\, x_{E|347}\, x_{E|7} \\ & + x_{7|E}\, x_{57|E}\, x_{4567|E}\, x_{E|23467}\, x_{E|36}\, x_{E|6} \\ & + x_{7|E}\, x_{67|E}\, x_{4567|E}\, x_{E|235}\, x_{E|35}\, x_{E|5}. \end{aligned}$$

This expression is deceivingly short. Carrying out this seemingly simple computation by hand is very tedious; if one were to do it by brute force, one would find that the number of terms of the canonical expansions of $\delta^0, \dots, \delta^6$ are the following:

	δ^0	δ^1	δ^2	δ^3	δ^4	δ^5	δ^6
number of monomials counted with multiplicities	1	29	352	658	383	69	3
number of distinct monomials	1	29	333	621	370	68	3

This example shows typical behavior: for small k the number of cones in the expansion of δ^k increases with k, but as k approaches $n-1$, increasingly many products $x_{\mathcal{F}|\mathcal{G}}\,\delta$ are zero, and the canonical expansions become shorter.

We summarize the properties of the canonical expansion in the following proposition, which follows readily from the previous discussion.

Proposition 4.4. For each $m \geqslant 0$, the canonical expansion of δ^m of Definition 4.1 is the sum of the monomials indexed by the collection $\mathcal{T}^m_{\mathrm{M},\mathrm{M}^\perp}$ of all the *tables* $(\mathcal{F}|\mathcal{G}, \mathbf{e})$ of M for which

(1) $\mathcal{F}|\mathcal{G}$ is a biflag of length m, and

(2) $\mathbf{e} = (e_1, \dots, e_m)$ is a sequence of distinct elements of E such that $e_i \in F_i \cap G_i$, and

$$e_i = \max\big(E - \bigcup_{j:\, e_j > e_i} (F_j \cap G_j)\big) \qquad \text{for all } 1 \leqslant i \leqslant m.$$

In symbols, the following identity holds in the Chow ring $A_{\mathrm{M},\mathrm{M}^\perp}$:

$$\delta^m = \sum_{(\mathcal{F}|\mathcal{G},\mathbf{e}) \in \mathcal{T}^m_{\mathrm{M},\mathrm{M}^\perp}} x_{F_1|G_1} x_{F_2|G_2} \cdots x_{F_m|G_m}.$$

We encode such a pair $(\mathcal{F}|\mathcal{G}, \mathbf{e})$ in the following table.

$$(\mathcal{F}|\mathcal{G},\mathbf{e}): \quad \begin{array}{|cc|ccccccccccccc|cc|}
\hline
\varnothing & \subsetneq & F_1 & \subseteq & \cdots & \subseteq & F_d & \subseteq & F_{d+1} & \subseteq & \cdots & \subseteq & F_m & \subseteq & E \\
E & \supseteq & G_1 & \supseteq & \cdots & \supseteq & G_d & \supseteq & G_{d+1} & \supseteq & \cdots & \supseteq & G_m & \supsetneq & \varnothing \\
\hline
& & e_1 & & \cdots & & e_d & & e_{d+1} & & \cdots & & e_m & & \\
\hline
\end{array}$$

We adopt the convention that $F_0 = G_{m+1} = \varnothing$ and $G_0 = F_{m+1} = E$.

As Example 4.3 illustrates, the canonical expansion of δ^m may contain repeated terms $x_{\mathcal{F}|\mathcal{G}}$ coming from tables that have the same biflag $\mathcal{F}|\mathcal{G}$ but different sequences $\mathbf{e}$.

Example 4.5. Let us revisit the canonical expansion of δ^6, the highest non-zero power of δ, in Example 4.3. The first monomial arises from the following table $(\mathcal{F}|\mathcal{G}, \mathbf{e})$:

$$\begin{array}{|cc|ccccccccccc|cc|}
\hline
\varnothing & \subset & \mathbf{6} & \subsetneq & \mathbf{5}6 & \subsetneq & \mathbf{4}567 & \subsetneq & E & = & E & = & E & = & E \\
E & = & E & = & E & = & E & \supsetneq & \mathbf{2}3467 & \supsetneq & \mathbf{3}47 & \supsetneq & \mathbf{7} & \supset & \varnothing \\
\hline
& & e_1 = 6 & & e_2 = 5 & & e_3 = 4 & & e_4 = 2 & & e_5 = 3 & & e_6 = 7 & & \\
\hline
\end{array}$$

The terms $x_{F_i|G_i}$ arrive to the monomial in the order $x_{E|\mathbf{7}} x_{\mathbf{6}|E} x_{\mathbf{5}6|E} x_{\mathbf{4}567|E} x_{E|\mathbf{3}47} x_{E|\mathbf{2}3467}$, in decreasing order of the e_is. The two other monomials are $x_{\mathbf{7}|E}\, x_{E|\mathbf{6}}\, x_{\mathbf{5}7|E}\, x_{\mathbf{4}567|E}\, x_{E|\mathbf{3}6}\, x_{E|\mathbf{2}3467}$ and $x_{\mathbf{7}|E}\, x_{\mathbf{6}7|E}\, x_{E|\mathbf{5}}\, x_{\mathbf{4}567|E}\, x_{E|\mathbf{3}5}\, x_{E|\mathbf{2}35}$, where the terms are again listed in order of arrival.

4.2. **The beta invariant of a matroid in its conormal intersection theory.** Our next goal is to prove Proposition 4.8, which describes the canonical expansion of δ^{n-1} in the Chow ring $A_{\mathrm{M},\mathrm{M}^\perp}$, and uses it to conclude that its degree is Crapo's beta invariant $\beta(\mathrm{M})$.

For each basis $B \subseteq E$ of M, denote the corresponding dual basis of $\mathrm{M}^\perp$ by

$$B^\perp \coloneqq E - B.$$

We also let $\mathrm{cl}^\perp$ denote the closure function of $\mathrm{M}^\perp$.

A *broken circuit* of M is a set of the form $C - \min C$ where C is a circuit. An $\mathbf{nbc}$-*basis* of M is a basis that contains no broken circuits. A $\beta\,\mathbf{nbc}$-*basis* of M is an $\mathbf{nbc}$ basis B of M such that $B^\perp \cup \{0\} - \{1\}$ is an $\mathbf{nbc}$ basis of $\mathrm{M}^\perp$.[16] The number of $\mathbf{nbc}$ basis is the Möbius number $|\mu(\mathrm{M})| = |\mu(\varnothing, E)|$, whereas the number of $\beta\,\mathbf{nbc}$ bases is the beta invariant $\beta(\mathrm{M})$ [Zie92].

It is well known that the independence complex $\mathrm{IN}(M)$ and the reduced broken circuit complex $\overline{\mathrm{BC}}(M)$ of a matroid M are shellable, and hence homotopy equivalent to wedges of spheres. The $\mathbf{nbc}$ bases and $\beta\,\mathbf{nbc}$ bases of M naturally index the spheres in the lexicographic shellings of $\mathrm{IN}(M)$ and $\overline{\mathrm{BC}}(M)$, respectively [Bjö92, Zie92].

Definition 4.6. Let B be a $\beta\,\mathbf{nbc}$ basis of M and write

$$B - 0 = \{e_1 > \cdots > e_r\}, \qquad\qquad B^\perp - 1 = \{e_{r+1} < \cdots < e_{n-1}\}.$$

The *maximal biflag* $\mathcal{F}(B)|\mathcal{G}(B)$ *and the* $\beta\,\mathrm{cone}(B) \coloneqq \sigma_{\mathcal{F}(B)|\mathcal{G}(B)}$ *of* B are

$$\begin{array}{|cc|ccccccccc|cc|}\hline \varnothing & \subsetneq & \mathrm{cl}_{\mathrm{M}}(e_1) & \subsetneq & \cdots & \subsetneq & \mathrm{cl}(e_1,\dots,e_r) & \subsetneq & E & = & \cdots & = & E & = & E \\ E & = & E & = & \cdots & = & E & \supsetneq & \mathrm{cl}^\perp(e_{r+1},\dots,e_{n-1}) & \supsetneq & \cdots & \supsetneq & \mathrm{cl}^\perp(e_{n-1}) & \supsetneq & \varnothing \\ \hline \end{array}$$

To see that $\mathcal{F}(B)|\mathcal{G}(B)$ is indeed a biflag, we verify that $\mathrm{cl}(B-0) \cup \mathrm{cl}^\perp(B^\perp - 1) \neq E$. Notice that $1 \notin \mathrm{cl}^\perp(B^\perp - 1)$ since $B^\perp$ is a basis of $\mathrm{M}^\perp$; and if we had $1 \in \mathrm{cl}(B-0)$, then $B - 0 \cup 1$ would contain a circuit C whose minimum element is 1, and hence B would contain the broken circuit $C - 1$, contradicting that B is $\mathbf{nbc}$.

Example 4.7. The matroid of Figure 6 has three $\beta\,\mathbf{nbc}$ basis, namely

$$B_1 = 0456, B_2 = 0457, B_3 = 0467.$$

The corresponding βcones are precisely the ones arising in the expansion of Example 4.3. The following theorem shows this is a general phenomenon.

Proposition 4.8. Let M be a loopless and coloopless matroid on the ground set $E = \{0,\dots,n\}$. Then, in the Chow ring of the conormal fan of M, we have the canonical expansion

$$\delta^{n-1} = \sum_{B \in \beta\,\mathbf{nbc}(\mathrm{M})} x_{\beta\,\mathrm{cone}(B)}.$$

It follows that the degree of δ^{n-1} is the β-invariant of M.

[16] This definition is different from the standard one, but they are readily proved to be equivalent.

Proof. We proceed in a series of lemmas. Proposition 4.4 describes the canonical expansion of δ^{n-1} in terms of tables $(\mathcal{F}|\mathcal{G}, \mathbf{e})$ of the form

$$\begin{array}{|cc|ccccccccccc|cc|}
\hline
\varnothing & \subsetneq & F_1 & \subseteq & \cdots & \subseteq & F_d & \subsetneq & F_{d+1} & \subseteq & \cdots & \subseteq & F_{n-1} & \subseteq & E \\
E & \supseteq & G_1 & \supseteq & \cdots & \supseteq & G_d & \supsetneq & G_{d+1} & \supseteq & \cdots & \supseteq & G_{n-1} & \supsetneq & \varnothing \\
\hline
 & & e_1 & & \cdots & & e_d & & e_{d+1} & & \cdots & & e_{n-1} & & \\
\hline
\end{array},$$

which have a unique double jump $d = j(\mathcal{F}) \cap j(\mathcal{G})$ thanks to Proposition 3.17. A priori, this double jump could occur at $d = 0$ or $d = n-1$. We let

$$\{e_n, e_{n+1}\} \coloneqq E - \{e_1, \ldots, e_{n-1}\}$$

be the two elements missing from the sequence $\mathbf{e}$. Let us record two simple observations about such tables, which we will return to often.

Lemma 4.9. If $i \in \mathsf{J}(\mathcal{F}) - \mathsf{J}(\mathcal{G})$, then $e_i > e_{i+1}$. If $i \in \mathsf{J}(\mathcal{G}) - \mathsf{J}(\mathcal{F})$, then $e_i < e_{i+1}$.

Proof. By symmetry, it suffices to prove the first assertion. Assume contrariwise that $i \in \mathsf{J}(\mathcal{F}) - \mathsf{J}(\mathcal{G})$ and $e_i < e_{i+1}$, so the table $(\mathcal{F}|\mathcal{G}, \mathbf{e})$ contains

$$\begin{array}{|ccc|}
\hline
F_i & \subsetneq & F_{i+1} \\
G_i & = & G_{i+1} \\
\hline
e_i & < & e_{i+1} \\
\hline
\end{array}.$$

Then the pair $F_i|G_i$ arrives to the monomial $x_{\mathcal{F}|\mathcal{G}}$ after $F_{i+1}|G_{i+1}$, so $e_i \notin F_{i+1} \cap G_{i+1}$. This contradicts that $e_i \in F_i \cap G_i \subseteq F_{i+1} \cap G_i = F_{i+1} \cap G_{i+1}$. □

Lemma 4.10. If $i < j$ and $e_i < e_j$, then $e_i \notin G_j$. If $i < j$ and $e_i > e_j$, then $e_j \notin F_i$.

Proof. It suffices to prove the first assertion. The table $(\mathcal{F}|\mathcal{G}, \mathbf{e})$ contains

$$\begin{array}{|ccccc|}
\hline
F_i & \subseteq & \cdots & \subseteq & F_j \\
G_i & \supseteq & \cdots & \supseteq & G_j \\
\hline
e_i & & < & & e_j \\
\hline
\end{array},$$

which shows that $F_i|G_i$ appears in the term $x_{\mathcal{F}|\mathcal{G}}$ after $F_j|G_j$, so $e_i \notin F_j \cap G_j$. Since $e_i \in F_i \subseteq F_j$, we must have $e_i \notin G_j$. □

Lemma 4.11. If the table $(\mathcal{F}|\mathcal{G}, \mathbf{e})$ arises in the canonical expansion of δ^{n-1}, then its unique double jump is at $d = r$, and its table is of the form

$$\begin{array}{|ccccccccccccccc|}
\hline
\varnothing & \subsetneq & F_1 & \subsetneq & \cdots & \subsetneq & F_r & \subsetneq & E & = & \cdots & = & E & = & E \\
E & = & E & = & \cdots & = & E & \supsetneq & G_{r+1} & \supsetneq & \cdots & \supsetneq & G_{n-1} & \supsetneq & \varnothing \\
\hline
 & & e_1 & > & \cdots & > & e_r & \{e_n, e_{n+1}\} & e_{r+1} & < & \cdots & < & e_{n-1} & & \\
\hline
\end{array}.$$

The unique nonempty gap is $D_r = \{e_n, e_{n+1}\}$; we write it under the double jump at r.

Proof. If d is the unique double jump, D_d is the unique nonempty gap. Since $\{e_1, \ldots, e_d\} \subseteq F_d$ and $\{e_{d+1}, \ldots, e_{n-1}\} \subseteq G_{d+1}$, we must have $F_d \cup G_{d+1} = \{e_1, \ldots, e_{n-1}\}$ by Lemma 3.15. Therefore the gap $D_d = E - (F_d \cup G_{d+1})$ indeed equals $\{e_n, e_{n+1}\}$.

Now we prove that

$$e_1 > e_2 > \cdots > e_d \qquad \text{and} \qquad e_{d+1} < \cdots < e_{n-2} < e_{n-1}.$$

By symmetry, it suffices to show the first claim. For contradiction, suppose that $e_j < e_{j+1}$ for a minimal choice of $j < d$. If $j > 1$ then $e_{j-1} > e_j$ implies $e_j \notin F_{j-1}$ by Lemma 4.9; if $j = 1$ this holds trivially. On the other hand $e_j < e_{j+1}$ implies $e_j \notin G_{j+1}$ by Lemma 4.10. However we have $\{e_1, \ldots, e_{j-1}\} \subseteq F_{j-1}$ and $\{e_{j+1}, \ldots, e_{n-1}\} \subseteq G_{j+1}$, and also $\{e_n, e_{n+1}\} \subseteq G_d \subseteq G_{j+1}$; therefore $F_{j-1} \cup G_{j+1} = E - e_j$. This contradicts Lemma 3.15, proving the first claim.

Now, for $j = 1, \ldots, d-1$, the inequality $e_j > e_{j+1}$ implies that $e_{j+1} \in F_{j+1} - F_j$ and hence $j \in \mathsf{J}(\mathcal{F})$. It follows that $\{0, 1, \ldots, d\} = \mathsf{J}(\mathcal{F})$ and similarly $\{d, \ldots, n-2, n-1\} = \mathsf{J}(\mathcal{G})$. Therefore $d = r$. Additionally, since $\mathsf{J}(\mathcal{G})$ does not contain $0, 1, \ldots, d-1$, we must have $E = G_1 = \cdots = G_d$, and similarly $F_{d+1} = \cdots = F_{n-1} = E$. $\square$

Lemma 4.12. If a table $(\mathcal{F}|\mathcal{G}, \mathbf{e})$ arises in the canonical expansion of δ^{n-1}, then $\{e_1, \ldots, e_{n-1}\} = \{2, 3, \ldots, n\}$ and $\{e_n, e_{n+1}\} = \{0, 1\}$. Moreover,

$$e_i = \min F_i \quad \text{for } 1 \leqslant i \leqslant r \qquad \text{and} \qquad e_i = \min G_i \quad \text{for } r+1 \leqslant i \leqslant n-1,$$

and

$$F_i = \mathrm{cl}(e_1, \ldots, e_i) \quad \text{for } 1 \leqslant i \leqslant r \qquad \text{and} \qquad G_i = \mathrm{cl}^{\perp}(e_i, \ldots, e_{n-1}) \quad \text{for } r+1 \leqslant i \leqslant n-1.$$

In particular, the sequence $\mathbf{e}$ and the biflag $\mathcal{F}|\mathcal{G}$ determine each other.

Proof. Let us assume without loss of generality that $e_r < e_{r+1}$, so x_{F_r, G_r} is the last term to arrive in the monomial corresponding to $(\mathcal{F}|\mathcal{G}, \mathbf{e})$. By definition,

$$e_r = \max \big(E - \bigcup_{\substack{1 \leqslant j \leqslant n-1 \\ j \neq r}} (F_j \cap G_j)\big) = \max \big(E - (F_{r-1} \cup G_{r+1})\big).$$

If we had $e_r \leqslant 1$, then $|F_{r-1} \cup G_{r+1}| \geqslant n-1$ which would imply $|F_r \cup G_{r+1}| = n$, a contradiction by Lemma 3.15. Thus $e_r = 2$ and the first claim follows. Also $F_r \cup G_{r+1} = E - \{0, 1\}$.

Now let us show $e_i = \min F_i$ for $1 \leqslant i \leqslant r$. If that were not the case, then since $0, 1 \notin F_i$, we would have $\min F_i = e_j < e_i$ for some $j \neq i$. Since $e_1 > \cdots > e_i$, this would imply $i < j$, and Lemma 4.10 would then tell us that $e_j \notin F_i$, a contradiction. Similarly $e_i = \min G_i$ for $r+1 \leqslant i \leqslant n-1$.

Finally, since $e_1 \in F_1,\ e_2 \in F_2 - F_1,\ \ldots,\ e_i \in F_i - F_{i-1}$ and F_i has rank i, the elements $e_1, \ldots, e_i$ must be independent and span F_i. The analogous result holds for G_i as well. $\square$

Lemma 4.13. If a table $(\mathcal{F}|\mathcal{G}, \mathbf{e})$ arises in the canonical expansion of δ^{n-1}, then $\{0, e_1, \ldots, e_r\}$ is a βnbc basis.

Proof. Since $e_r = \min F_r$, we have $0 \notin F_r = \mathrm{cl}(e_1, \ldots, e_r)$. Therefore $B = \{0, e_1, \ldots, e_r\}$ is indeed a basis. We prove that B is **nbc** by contradiction; assume that it contains a broken circuit $C - \min C$. Since $\min C \notin B$, there are two cases:

(i) $\min C = 1$. Let $C = \{1, e_{a_1}, \ldots, e_{a_k}\}$ where $1 \leqslant a_1 < \ldots < a_k \leqslant r$. Then $1 \in \mathrm{cl}(e_{a_1}, \ldots, e_{a_k}) \subseteq F_{a_k} \subseteq F_r$. This contradicts that $\{0, 1\} = E - (F_r \cup G_{r+1})$.

(ii) $\min C = e_s$ for some $s \geqslant r + 1$. Let $C = \{e_s, e_{a_1}, \ldots, e_{a_k}\}$ where $1 \leqslant a_1 < \ldots < a_k \leqslant r$. Then $e_s \in \mathrm{cl}(e_{a_1}, \ldots, e_{a_k}) \subseteq F_{a_k}$. This contradicts Lemma 4.10 since $a_k \leqslant r < s$ and $e_{a_k} > e_s$.

An analogous argument shows that $B^\perp - \{0\} \cup \{1\} = \{0, e_{r+1}, \ldots, e_{n-1}\}$ is an **nbc** basis of $\mathrm{M}^\perp$. We conclude that B is β **nbc**, as desired. □

We now have all the ingredients to complete the proof of Proposition 4.8.

Lemma 4.12 tells us that each monomial $x_{\mathcal{F}|\mathcal{G}}$ that appears in the canonical expansion of δ^{n-1} has coefficient $+1$. Combined with Lemma 4.13, it also tells us that every term that appears is of the form $x_{\beta\,\mathrm{cone}(B)}$ for a β **nbc** basis B.

Conversely, if $\mathcal{F}|\mathcal{G} = \mathcal{F}(B)|\mathcal{G}(B)$ is the biflag of a β **nbc** basis B, and if we define $\mathbf{e}$ by setting $B = \{e_1 > \cdots > e_r > 0\}$ and $E - B = \{e_{n-1} > \cdots > e_{r+1} > 1\}$, then it is straightforward to check that the table $(\mathcal{F}|\mathcal{G}, \mathbf{e})$ satisfies the conditions of Proposition 4.4, so it does in fact arise in the canonical expansion of δ^{n-1}.

This proves the formula for δ^{n-1} and for its degree, in light of Definition 3.4. □

4.3. **A vanishing lemma.** Throughout the remainder of this section, we fix a strictly increasing flag of nontrivial flats

$$\mathcal{F} = \{F_1 \subsetneq \cdots \subsetneq F_k\},$$

following the convention that $F_0 = \varnothing$ and $F_{k+1} = E$. We define the *orthogonal flag* $\mathcal{F}^\perp$ of flats of $\mathrm{M}^\perp$ by

$$\mathcal{F}^\perp = \{F_1^\perp \supseteq \cdots \supseteq F_k^\perp\}, \quad \text{where } F_i^\perp = \mathrm{cl}^\perp(E - F_i) \text{ for } 1 \leqslant i \leqslant k.$$

Note that the flag $\mathcal{F}^\perp$ may contain repeated coflats, and it may contain the trivial coflat E. We call the interval $[F_{i-1}, F_i]$ *short* if $|F_i - F_{i-1}| = 1$ and *long* otherwise. Recall that we denote the corresponding minor by $\mathrm{M}(i) \coloneqq \mathrm{M}[F_{i-1}, F_i]$.

The following lemma shows that many monomials in the Chow ring $A_{\mathrm{M},\mathrm{M}^\perp}$ vanish when multiplied by the highest possible power of δ.

Lemma 4.14. (Vanishing Lemma) Let $\mathcal{F}|\mathcal{G}$ be a biflag of M of length k such that $\mathcal{F}$ is strictly increasing, and suppose that

$$x_{\mathcal{F}|\mathcal{G}}\, \delta^{n-k-1} \neq 0$$

in the Chow ring of the conormal fan of M. Then

(1) $\mathcal{G} = \mathcal{F}^\perp$, and

(2) every long interval $\mathrm{M}(i) = \mathrm{M}[F_{i-1}, F_i]$ for $1 \leqslant i \leqslant k+1$ is loopless and coloopless.

Proof. Let us assume $x_{\mathcal{F}|\mathcal{G}}\, \delta^{n-k-1} \neq 0$ and consider a non-zero term $x_{\mathcal{F}^+|\mathcal{G}^+}$ arising in the canonical expansion of $x_{\mathcal{F}|\mathcal{G}}\, \delta^{n-k-1}$. Let

$$\mathcal{F}|\mathcal{G} = \mathcal{F}_k|\mathcal{G}_k,\ \mathcal{F}_{k+1}|\mathcal{G}_{k+1},\ \ldots,\ \mathcal{F}_{n-1}|\mathcal{G}_{n-1} = \mathcal{F}^+|\mathcal{G}^+$$

be some sequence of biflags obtained by recursively applying Lemma 4.2 to this expansion. For $k \leqslant i \leqslant n-1$, the biflag $\sigma_{\mathcal{F}_i|\mathcal{G}_i}$ has i rays. Let $D_{i,0}|\cdots|D_{i,i}$ be its sequence of gaps as described in Definition 3.10. With Lemma 3.13 in mind, let

$$Y_i = \bigsqcup_{j=0}^{i} D_{i,j} = E - \bigcup_{F|G \in \mathcal{F}_i|\mathcal{G}_i} (F \cap G) \tag{4.3.1}$$

be the union of the gaps in the biflag $\mathcal{F}_i|\mathcal{G}_i$. In particular, $D_{k,0}|\cdots|D_{k,k} = D_0|\cdots|D_k$ and $Y_k = Y$ are the gap sequence and the union of the gaps of the initial flag $\mathcal{F}|\mathcal{G}$. To prove the Vanishing Lemma 4.14 we need a preliminary lemma.

Lemma 4.15. Suppose the conditions of Lemma 4.14 hold. Then

(1) If $\mathcal{F}|\mathcal{G}$ has z empty gaps, then the union Y of its gaps has size $|Y| = n+1-z$.

(2) For each empty gap $D_l = \varnothing$ we have $F_{l+1} - F_l = \{e_l\}$ for some $e_l \in E$. Furthermore, the union of the gaps is $Y = E - \{e_l \,:\, D_l = \varnothing\}$.

(3) For all $0 \leqslant i \leqslant k$ we have

$$|F_{i+1} - F_i| = (r_{i+1} - r_i) + (r_i^\perp - r_{i+1}^\perp) \tag{4.3.2}$$

where we denote $r_j = r_{\mathrm{M}}(F_j)$ and $r_j^\perp = r_{\mathrm{M}^\perp}(G_j)$.

Proof of Lemma 4.15. 1. First let us prove that

$$|Y| \leqslant n+1-z. \tag{4.3.3}$$

For each empty gap $D_l = \varnothing$, choose an element $e_l \in F_{l+1} - F_l$. Since $e_l \notin D_l = E - (F_l \cup G_{l+1})$, we must have $e_l \in G_{l+1}$. This implies that $e_l \in F_{l+1} \cap G_{l+1}$, so (4.3.1) gives $e_l \notin Y$. There are z such elements e_l, which are all distinct by construction; this implies (4.3.3).

Now let us prove the opposite inequality

$$|Y| \geqslant n+1-z. \tag{4.3.4}$$

We obtain $\mathcal{F}_{i+1}|\mathcal{G}_{i+1}$ from $\mathcal{F}_i|\mathcal{G}_i$ by choosing the largest gap element $e = \max Y_i$, finding the unique gap $D_{i,j}$ of $\mathcal{F}_i|\mathcal{G}_i$ containing e, and inserting a new pair $F|G$ with $e \in F \cap G$ between the jth and $(j+1)$th rays of $\mathcal{F}_i|\mathcal{G}_i$, as follows:

$$\mathcal{F}_{i+1}|\mathcal{G}_{i+1} : \boxed{\begin{array}{ccccccccc} \cdots & \subseteq & F_{i,j} & \subseteq & F & \subseteq & F_{i,j+1} & \subseteq & \cdots \\ \cdots & \supseteq & G_{i,j} & \supseteq & G & \supseteq & G_{i,j+1} & \supseteq & \cdots \end{array}}$$

Thus the only change between the gaps of $\mathcal{F}_i|\mathcal{G}_i$ and $\mathcal{F}_{i+1}|\mathcal{G}_{i+1}$ is that we are replacing the gap $D_{i,j}$ with two smaller disjoint gaps $D_{i+1,j}$ and $D_{i+1,j+1}$ that do not contain e:

$$D_{i,j} = E - (F_{i,j} \cup (G_{i,j+1}) \quad \longmapsto \quad \begin{cases} D_{i+1,j} = E - ((F_{i,j} \cup G) \\ D_{i+1,j+1} = E - (F \cup (G_{i+1,j+1}). \end{cases}$$

We have

$$D_{i,j} \supseteq D_{i+1,j} \sqcup D_{i+1,j+1} \sqcup e. \tag{4.3.5}$$

In the end, the final biflag $\mathcal{F}_{n-1}|\mathcal{G}_{n-1}$ has n gaps, of which $n-1$ are empty and one of them, say D, has size at least 2.

It is helpful to visualize this data as a graded forest of levels $k, k+1, \ldots, n-1$. The vertices of the top level k are the gaps $D_0, \ldots, D_k$ of the original biflag $\mathcal{F}_i|\mathcal{G}_i$; they are the roots of the $k+1$ trees in the forest. The vertices of the ith level are the gaps $D_{i,0}, \ldots, D_{i,i}$ of $\mathcal{F}_i|\mathcal{G}_i$. To go from level i to level $i+1$, we connect the split gap $D_{i,j}$ with the gaps $D_{i+1,j}$ and $D_{i+1,j+1}$ that replace it. Every other gap $D_{i,k}$ is connected to the gap in the next level that is equal to it; this is $D_{i+1,k}$ if $k < j$ and $D_{i+1,k+1}$ if $k > j$.

Each gap of $\mathcal{F}^+|\mathcal{G}^+ = \mathcal{F}_{n-1}|\mathcal{G}_{n-1}$, at the bottom level of the tree, descends from one of the original gaps of $\mathcal{F}|\mathcal{G} = \mathcal{F}_k|\mathcal{G}_k$ through successive gap replacements. Let

$$d_l = \text{number of gaps of } \mathcal{F}^+|\mathcal{G}^+ \text{ that descend from the initial gap } D_l \text{ of } \mathcal{F}|\mathcal{G},$$

for $0 \leqslant l \leqslant k$. We consider three cases:

Case 1. $D_l = \varnothing$:

In this case the gap D_l eventually becomes a single empty gap in $\mathcal{F}^+|\mathcal{G}^+$, so $d_l = 1$.

Case 2. $D_l \neq \varnothing$ is the progenitor of the unique non-empty gap D of $\mathcal{F}^+|\mathcal{G}^+$:

Consider the gaps that descend from D_l throughout this process. By (4.3.5), every time one such gap gets replaced by two smaller ones, the size of the union of the gaps strictly decreases. In the end this union has size $|D| \geqslant 2$. Therefore these gaps were split at most $|D_l| - 2$ times, so $d_l \leqslant |D_l| - 1$.

Case 3. $D_l \neq \varnothing$ is not the progenitor of the non-empty gap D:

Again, every time a descendant of D_l gets replaced by two smaller ones, the size of their union decreases. Furthermore, their union can never have size 1 by Proposition 3.14. Thus $d_l \leqslant |D_l|$.

Since the final number of gaps is n, we conclude that

$$n = \sum_{l=0}^{k} d_l \leqslant z + \left(\sum_{l\,:\,D_l \neq \varnothing} |D_l| \right) - 1 = z + |Y| - 1,$$

where z is the number of empty gaps D_l. This proves (4.3.4).

Since the two opposite inequalities (4.3.3) and (4.3.4) and hold, we must have

$$|Y| = n + 1 - z,$$

proving part 1. of the lemma. Furthermore, every inequality we applied along the way must in fact have been an equality. Let us record these:

a) For (4.3.3) to be an equality, we must have $F_{l+1} - F_l = \{e_l\}$ for each empty gap $D_l = \varnothing$, and $Y = E - \{e_l : D_l = \varnothing\}$.

b) For (4.3.4) to be an equality, we must have

$$d_l = 1 \text{ in case } 1, \quad d_l = |D_l| - 1 \text{ in case } 2, \quad d_l = |D_l| \text{ in case } 3. \tag{4.3.6}$$

We use this to prove (4.3.2), in two steps. First we prove that

$$d_l = \begin{cases} |F_{l+1} - F_l| & \text{if } D_l \text{ is in case 1 or 3 above,} \\ |F_{l+1} - F_l| - 1 & \text{if } D_l \text{ is in case 2.} \end{cases} \tag{4.3.7}$$

<u>Case 1.</u> $D_l = \varnothing$:

In this case we have $d_l = 1$, and $|F_{l+1} - F_l| = 1$ by a).

<u>Cases 2 and 3.</u> $D_l \neq \varnothing$.

We claim that

$$D_l = F_{l+1} - F_l \tag{4.3.8}$$

which will imply the claim by b). The forward inclusion holds by definition. For the backward inclusion, consider $e \in F_{l+1} - F_l$. By a) we must have $e \in Y$ and since D_l is the only gap intersecting $F_{l+1} - F_l$, we must have $e \in D_l$.

Next we prove that

$$d_l = \begin{cases} (r_{l+1} - r_l) + (r_l^\perp - r_{l+1}^\perp) & \text{if } D_l \text{ is in case 1 or 3} \\ (r_{l+1} - r_l) + (r_l^\perp - r_{l+1}^\perp) - 1 & \text{if } D_l \text{ is in case 2.} \end{cases} \tag{4.3.9}$$

<u>Case 1 and 3.</u> D_l is not the progenitor of the double gap D:

In these cases, the part of $\mathcal{F}^+|\mathcal{G}^+$ between $F_l|G_l$ and $F_{l+1}|G_{l+1}$ contains no double jumps. In each of the d_l single jumps, either the rank increases by 1 or the corank decreases by 1, but not both. Therefore d_l must equal the sum of the rank increase $r_{l+1} - r_l$ and the corank decrease $r_l^\perp - r_{l+1}^\perp$.

<u>Case 2.</u> D_l is the progenitor of the double gap D:

In these cases, the part of $\mathcal{F}^+|\mathcal{G}^+$ between $F_l|G_l$ and $F_{l+1}|G_{l+1}$ contains one double jump. In each of the $d_l - 1$ single jumps, either the rank increases by 1 or the corank decreases by 1, but not both. In the double jump, both changes occur. Therefore $d_l + 1$ must equal the sum of the rank increase $r_{l+1} - r_l$ and the corank decrease $r_l^\perp - r_{l+1}^\perp$.

The desired result now follows from (4.3.7) and (4.3.9). □

With Lemma 4.15 at hand, we are finally ready to prove the Vanishing Lemma 4.14.

First, we prove that $\mathcal{G} = \mathcal{F}^\perp$. One readily verifies, using the rank function of the dual matroid, that

$$(r_{i+1} - r_i) + (r_{\mathrm{M}^\perp}(E - F_i) - r_{\mathrm{M}^\perp}(E - F_{i+1})) = |F_{i+1} - F_i| \qquad \text{for } 0 \leqslant i \leqslant k.$$

By Lemma 4.15(3), the sequences $(r_{\mathrm{M}^\perp}(E - F_i) \,:\, 0 \leqslant i \leqslant k)$ and $(r_i^\perp \,:\, 0 \leqslant i \leqslant k)$ satisfy the same recurrence; they also have the same initial value $r_{\mathrm{M}^\perp}(E - F_0) = r^\perp = r_0^\perp$ since $F_0 = \varnothing$ and $G_0 = E$. We conclude that

$$r_{\mathrm{M}^\perp}(E - F_i) = r_{\mathrm{M}^\perp}(G_i) \qquad \text{for } 0 \leqslant i \leqslant k.$$

But $F_i \cup G_i = E$ implies $G_i \supseteq E - F_i$, and since G_i is a coflat, $G_i \supseteq \mathrm{cl}^\perp(E - F_i) = F_i^\perp$. It follows that $G_i \supseteq F_i^\perp$ are flats of the same rank in $\mathrm{M}^\perp$, so $G_i = F_i^\perp$ for all i as desired.

Next, we prove that every long interval $\mathrm{M}(i) = \mathrm{M}[F_{i-1}, F_i]$ is loopless and coloopless. We proceed by contradiction.

First assume that $\mathrm{M}(i) = (\mathrm{M}/F_{i-1})|(F_i - F_{i-1})$ has a loop l. Since restriction cannot create new loops, l must also be a loop of M/F_{i-1}. This contradicts the fact that F_{i-1} is a flat.

Now assume that $\mathrm{M}(i) = (\mathrm{M}|F_i)/F_{i-1}$ has a coloop c. Since contraction cannot create new coloops, c must also be a coloop of $\mathrm{M}|F_i$. Thus $r_{\mathrm{M}}(F_i - c) = r_{\mathrm{M}}(F_i) - 1$, which implies that $r_{\mathrm{M}^\perp}((E - F_i) \cup c) = r_{\mathrm{M}^\perp}(E - F_i)$. This means that $c \in \mathrm{cl}^\perp(E - F_i) = F_i^\perp$.

Now, since $\mathrm{M}(i)$ is long, Lemma 4.15(2) implies that $D_i \neq \varnothing$ and that $c \in Y$. But then we must have $c \in D_i = (F_i - F_{i-1}) \cap (F_{i-1}^\perp - F_i^\perp)$, contradicting that $c \in F_i^\perp$. The desired result follows. □

4.4. **The beta invariant of a flag in its conormal intersection theory.** In this section we complete the proof that $\deg(\pi^*(x_{\mathcal{F}})\delta^{n-k-1}) = \beta(\mathrm{M}[\mathcal{F}])$ for any strictly increasing flag of flats $\mathcal{F}$ in M of length k. We will first need a lemma relating the conormal fan of M with that of the deletion M/i.

Let i be an arbitrary element of E; recall that M has no coloops, so $i^\perp = E$ and $i|E$ is a biflat of M. The ambient space of $\mathrm{st}_{i|E}\,\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ is $(N_E \oplus N_E)/(\mathbf{e}_i + \mathbf{f}_E) = (N_E/\mathbf{e}_i) \oplus N_E$. We let $\overline{\mathbf{e}}_S$ be the image of $\mathbf{e}_S$ in $N_E/\mathbf{e}_i$ for $S \subseteq E$. We also let $\overline{x}_{F|G}$ be the variable in the Chow ring of the star corresponding to a ray $F|G$; we set it equal to 0 if $F|G$ is not a ray in this star.

Lemma 4.16. Consider the natural projection $\psi : (N_E/\mathbf{e}_i) \oplus N_E \longrightarrow N_{E-i} \oplus N_{E-i}$.

(1) The projection ψ induces a morphism of fans

$$\psi : \mathrm{st}_{i|E}\,\Sigma_{\mathrm{M},\mathrm{M}^\perp} \longrightarrow \Sigma_{\mathrm{M}/i,(\mathrm{M}/i)^\perp}.$$

(2) The corresponding pullback of Chow rings $\psi^* : A_{\mathrm{M}/i,(\mathrm{M}/i)^*} \to A(\mathrm{st}_{i|E}\,\Sigma_{\mathrm{M},\mathrm{M}^\perp})$ is given by

$$\psi^*(x_{A|B}) = \overline{x}_{(A\cup i)|B} + \overline{x}_{(A\cup i)|(B\cup i)}, \qquad A|B \text{ biflat of } \mathrm{M}/i,$$

where at least one of the terms in the right hand side is nonzero.

(3) The pullback ψ^* maps the class δ of $A_{\mathrm{M}/i,(\mathrm{M}/i)^\perp}$ to the following class of $A(\mathrm{st}_{i|E}\,\Sigma_{\mathrm{M},\mathrm{M}^\perp})$:

$$\overline{\delta} \coloneqq \psi^*(\delta) = \sum_{\substack{F|G \text{ biflat of } \mathrm{M} \\ i\in F,\ j\in F,G}} \overline{x}_{F|G} \qquad \text{for any } j \in E.$$

(4) The pullback ψ^* commutes with the degree maps of $A_{\mathrm{M}/i,(\mathrm{M}/i)^\perp}$ and $A(\mathrm{st}_{i|E}\,\Sigma_{\mathrm{M},\mathrm{M}^\perp})$; that is, $\deg_{\mathrm{M}/i}\eta = \deg_{\mathrm{st}}(\psi^*\eta)$ for all $\eta \in A^{n-2}_{\mathrm{M}/i,(\mathrm{M}/i)^\perp}$.

Proof. 1. The image of a ray $F|G$ in the star is

$$\psi(\overline{\mathbf{e}}_F + \mathbf{f}_G) = \mathbf{e}_{F-i} + \mathbf{f}_{G-i}, \qquad F|G \text{ biflat}, \ i \in F,$$

which is a ray of the conormal fan $\Sigma_{\mathrm{M}/i,(\mathrm{M}/i)^\perp}$ because $(F-i)|(G-i)$ is a biflat of M/i:

$$\begin{aligned} \mathrm{cl}_{\mathrm{M}/i}(F-i) &= \mathrm{cl}_{\mathrm{M}}(F) - i = F - i, \text{ and} \\ \mathrm{cl}_{\mathrm{M}^\perp - i}(G-i) &= \mathrm{cl}_{\mathrm{M}^\perp}(G-i) - i \subseteq \mathrm{cl}_{\mathrm{M}^\perp}(G) - i = G - i. \end{aligned}$$

Furthermore, if $i|E \cup \mathcal{F}|\mathcal{G}$ is a biflag of M, its gaps occur to the right of $i|E$, and there will also be gaps in the corresponding positions of $(\mathcal{F}-i)|(\mathcal{G}-i) \coloneqq \{(F-i)|(G-i) \,:\, F|G \in \mathcal{F}|\mathcal{G}\}$; so this will be a biflag of M/i. Therefore ψ maps cones to cones.

2. The value of the piecewise linear function $\psi^* x_{A|B}$ on a ray $\overline{\mathbf{e}}_F + \mathbf{f}_G$ of the star is

$$\psi^* x_{A|B}(\overline{\mathbf{e}}_F + \mathbf{f}_G) = x_{A|B}(\mathbf{e}_{F-i} + \mathbf{f}_{G-i})) = \begin{cases} 1 & \text{if } F = A\cup i \text{ and } G \in \{B, B\cup i\}, \text{ or} \\ 0 & \text{otherwise,} \end{cases}$$

taking into account that we must have $i \in F$. The fact that B is a flat of M/i implies that $\mathrm{cl}(B) \in \{B, B\cup i\}$, so at least one of the summands is nonzero.

3. We have

$$\psi^*(\delta_j) = \sum_{\substack{A|B \text{ biflat of } \mathrm{M}/i \\ j\in A,B}} (\overline{x}_{(A\cup i)|B} + \overline{x}_{(A\cup i)|(B\cup i)}) = \sum_{\substack{F|G \text{ biflat of } \mathrm{M} \\ i\in F,\ j\in F,G}} \overline{x}_{F|G} = \overline{\delta}_j.$$

4. We need to verify that

$$\deg_{\mathrm{M}/i} x_{\mathcal{A}|\mathcal{B}} = \deg_{\mathrm{st}} \psi^*(x_{\mathcal{A}|\mathcal{B}}) \coloneqq \deg_{\mathrm{M}}(x_{i|E}\,\psi^*(x_{\mathcal{A}|\mathcal{B}}))$$

for any maximal biflag $\mathcal{A}|\mathcal{B}$ of M/i. Writing $\psi^*(x_{A|B}) = \overline{x}_{(A\cup i)|B} + \overline{x}_{(A\cup i)|(B\cup i)}$ for each $A|B$ in $\mathcal{A}|\mathcal{B}$, we express $x_{i|E}\,\psi^*(x_{\mathcal{A}|\mathcal{B}})$ as a sum of squarefree monomials. One of the terms in this expression is $x_{i|E}\,x_{(\mathcal{A}\cup i)|\,\mathrm{cl}^\perp(\mathcal{B})}$, where $(\mathcal{A}\cup i)|\,\mathrm{cl}^\perp\mathcal{B} \coloneqq \{(A\cup i)|\,\mathrm{cl}^\perp(B)) \,:\, A|B \in \mathcal{A}|\mathcal{B}\}$. We need to prove this is the only nonzero term.

Consider any term $x_{i|E}x_{\mathcal{A}'|\mathcal{B}'}$ that arises in this expression. We automatically have $A'_j = A_j \cup i$ for all j, so it remains to prove $B'_j = \mathrm{cl}^\perp(B_j)$ for all j as well.

Let k be the largest index such that $i \in \mathrm{cl}^\perp(B_k)$. Then $\mathrm{cl}^\perp(B_j) = B_j \cup i$ for $j \leqslant k$ whereas $\mathrm{cl}^\perp(B_j) = B_j$ for $j \geqslant k+1$. For $1 \leqslant j \leqslant k$, B_j is not a flat in $\mathrm{M}^\perp$, so $B'_j = B_j \cup i = \mathrm{cl}^\perp(B_j)$

Now, notice that B_k and B_{k+1} are flats of consecutive ranks in $(\mathrm{M}/i)^\perp = \mathrm{M}^\perp - i$, so the flats $B_k \cup i = \mathrm{cl}^\perp(B_k)$ and $B_{k+1} = \mathrm{cl}^\perp(B_{k+1})$ of $\mathrm{M}^\perp$ also have consecutive ranks. Therefore $B_{k+1} \cup i$, which is strictly between them, cannot be a flat. Thus we must have $B'_{k+1} = B_{k+1}$, and hence $B'_j = B_j = \mathrm{cl}^\perp(B_j)$ for $j \geqslant k+1$ as well. We conclude that $\mathcal{A}'|\mathcal{B}' = (\mathcal{A} \cup i)|\,\mathrm{cl}^\perp(\mathcal{B})$ as desired. ☐

Now we can give an intersection-theoretic interpretation of the beta invariant of a flag.

Proposition 4.17. Let $\mathcal{F} = \{F_1 \subsetneq \cdots \subsetneq F_k\}$ be a strictly increasing flag of flats of M. We have

$$\deg(x_{\mathcal{F}|\mathcal{F}^\perp}\, \delta^{n-k-1}) = \beta(\mathrm{M}[\mathcal{F}]).$$

Proof. We proceed by induction on k. The case $k = 0$ is Proposition 4.8. For $k \geqslant 1$, let $F|F^\perp$ be the first biflat in $\mathcal{F}|\mathcal{F}^\perp$, and write $\mathcal{F}|\mathcal{F}^\perp = F|F^\perp \cup \mathcal{G}|\mathcal{G}^\perp$. Then $\mathcal{G} - F \coloneqq \{G - F : G \in \mathcal{G}\}$ is a flag of flats in M/F. It leads to the flag of biflats of M/F

$$(\mathcal{G} - F)|(\mathcal{G} - F)^\perp \coloneqq \{(G - F)|(G^\perp - F) : G \in \mathcal{G}\},$$

where the notation is justified by the fact that $G^\perp - F = \mathrm{cl}_{(\mathrm{M}/F)^\perp}((E - F) - (G - F))$ for $G \supseteq F$. We have

$$\beta(\mathrm{M}[\mathcal{F}]) = \beta(\mathrm{M}|F) \cdot \beta((\mathrm{M}/F)[\mathcal{G} - F])$$

because $\mathrm{M}[G_{j-1}, G_j] \cong (\mathrm{M}/F)[G_{j-1} - F,\, G_j - F]$ for $j = 1, \dots, k-1$. We consider two cases:

<u>Case 1.</u> $F = \{i\}$ for some $i \in E$.

Since $\beta(\mathrm{M}[\varnothing, i]) = 1$, we have

$$\begin{aligned}
\beta(\mathrm{M}[\mathcal{F}]) &= \beta((\mathrm{M}/i)[\mathcal{G} - i]) \\
&= \deg_{\mathrm{M}/i}\left(x_{\mathcal{G}-i|(\mathcal{G}-i)^\perp}\, \delta_{\mathrm{M}/i}^{(n-1)-(k-1)-1}\right) && \text{by the inductive hypothesis} \\
&= \deg_{\mathrm{st}}\left(\psi^*(x_{\mathcal{G}-i|(\mathcal{G}-i)^\perp})\, \overline{\delta}^{n-k-1}\right) && \text{by Lemma 4.16(3) and (4)} \\
&= \deg_{\mathrm{M}}\left(x_{i|E}\, \psi^*(x_{\mathcal{G}-i|(\mathcal{G}-i)^\perp})\, \delta_{\mathrm{M}}^{n-k-1})\right) && \text{since } x_{i|E}x_{F',G'} = 0 \text{ for } i \notin F' \\
&= \deg_{\mathrm{M}}\left(x_{i|E} \prod_{G \in \mathcal{G}} (x_{G|(G^\perp - i)} + x_{G|G^\perp})\, \delta_{\mathrm{M}}^{n-k-1}\right) && \text{by Lemma 4.16.2} \\
&= \deg_{\mathrm{M}}\left(x_{i|E}\, x_{\mathcal{G}|\mathcal{G}^\perp}\, \delta_{\mathrm{M}}^{n-k-1}\right) && \text{by the Vanishing Lemma 4.14} \\
&= \deg_{\mathrm{M}}\left(x_{\mathcal{F}|\mathcal{F}^\perp}\, \delta_{\mathrm{M}}^{n-k-1}\right).
\end{aligned}$$

<u>Case 2.</u> $|F| > 1$.

By the Vanishing Lemma 4.14, we may assume the interval $[\varnothing, F]$ is coloopless. This means that the flat F is *cyclic*; that is, $E - F$ is a coflat, and $F^\perp = E - F$. Then we have bijections

$$\begin{aligned}\phi_1 : \{\text{biflats of } \mathrm{M}\,|F\} &\longrightarrow \{\text{biflats } F'|G' \text{ of } \mathrm{M} \text{ with } F' \subseteq F \text{ and } G' \supseteq E - F\}\\ \phi_2 : \{\text{biflats of } \mathrm{M}\,/F\} &\longrightarrow \{\text{biflats } F'|G' \text{ of } \mathrm{M} \text{ with } F' \supseteq F \text{ and } G' \subseteq E - F\}\end{aligned}$$

given by $\phi_1(A|B) = A|(B \cup (E - F))$ and $\phi_2(A|B) = (A \cup F)|B$. These extend to bijections ϕ_1 (resp. ϕ_2) between the biflags of $\mathrm{M}\,|F$ (resp. $\mathrm{M}\,/F$) and the biflags of M that are supported on the corresponding set of biflats, and have a gap to the left (resp. to the right) of $F|(E - F)$.

Now let us compute $\deg(x_{\mathcal{F}|\mathcal{F}^\perp}\, \delta^{n-k-1})$ using the following variant of the canonical expansion of Definition 4.1, which proceeds in two stages:

Stage 1. At each step, choose e to be the largest gap element that is in F, if there is one.

Stage 2. At each step, choose e to be the largest gap element in $E - F$.

The first $|F| - 2$ steps of this computation will give $x_{\mathcal{F}|\mathcal{F}^\perp}$ times the image under ϕ_1 of the canonical expansion of $\delta_{M|F}^{|F|-2}$. By Proposition 4.8, there will be $\beta(\mathrm{M}\,|F)$ squarefree monomials.

Each such monomial will have a unique non-empty gap before F; say it is D_j, between biflats $F_j|G_j$ and $F_{j+1}|G_{j+1}$ of M, where F_j and F_{j+1} (resp. G_j and G_{j+1}) have consecutive ranks (resp. coranks). In step $|F| - 1$ of the computation, this gap D_j will be filled in a unique way by the biflat $F_{j+1}|G_j$. There will no longer be gap elements in F.

In step $|F|$, the computation will enter Stage 2 for each of the resulting $\beta(\mathrm{M}\,|F)$ monomials. The following $(|E-F|-1)-(k-1)-1$ steps will compute the image under ϕ_2 of the canonical expansion of $x_{(\mathcal{G}-F)|(\mathcal{G}-F)^\perp}\delta_{M/F}^{|F|-2}$. This expansion has $\beta((\mathrm{M}\,/F)[\mathcal{G} - F])$ squarefree monomials, by the inductive hypothesis.

Since $[|F|-2]+1+[(|E-F|-1)-(k-1)-1] = n-k-1$, this will conclude the computation of $x_{\mathcal{F}|\mathcal{F}^\perp}\delta^{n-k-1}$. The result will be the sum of $\beta(\mathrm{M}\,|F)\beta((\mathrm{M}\,/F)[\mathcal{G} - F]) = \beta(\mathrm{M}[\mathcal{F}])$ squarefree monomials, as we wished to prove. $\square$

Proposition 4.18. Let $\mathcal{F} = \{F_1 \subsetneq \cdots \subsetneq F_k\}$ be a strictly increasing flag of flats of M. We have

$$\deg(\pi^*(x_{\mathcal{F}})\, \delta^{n-k-1}) = \beta(\mathrm{M}[\mathcal{F}]).$$

Proof. Since $\pi^*(x_{\mathcal{F}}) = \sum_{\mathcal{F}|\mathcal{G} \text{ biflag}} x_{\mathcal{F}|\mathcal{G}}$, this follows from Lemma 4.14 and Proposition 4.17. $\square$

5. A CONORMAL INTERPRETATION OF THE CHERN–SCHWARTZ–MACPHERSON CYCLES

Recall that the k-dimensional Chern–Schwartz–MacPherson cycle of M is the Minkowski weight $\mathrm{csm}_k(\mathrm{M})$ on the Bergman fan of M defined by the formula

$$\mathrm{csm}_k(\mathrm{M})(\sigma_{\mathcal{F}}) = (-1)^{r-k}\beta(\mathrm{M}[\mathcal{F}]),$$

where $\sigma_{\mathcal{F}}$ is the k-dimensional cone corresponding to a flag of flats $\mathcal{F}$ of M.

Theorem 1.1. When M has no loops and no coloops, for every nonnegative integer $k \leqslant r$,

$$\mathrm{csm}_k(\mathrm{M}) = (-1)^{r-k}\pi_*(\delta^{n-k-1} \cap 1_{\mathrm{M},\mathrm{M}^\perp}).$$

Proof. We have

$$\begin{aligned}
\beta(\mathrm{M}[\mathcal{F}]) &= \deg_{\Sigma_{\mathrm{M},\mathrm{M}^\perp}}(\pi^*(x_{\mathcal{F}})\delta^{n-k-1}) && \text{by Proposition 4.18,}\\
&= (\pi^*(x_{\mathcal{F}})\delta^{n-k-1}) \cap 1_{\mathrm{M},\mathrm{M}^\perp} && \text{by Definition 3.4,}\\
&= x_{\mathcal{F}} \cap \pi_*(\delta^{n-k-1} \cap 1_{\mathrm{M},\mathrm{M}^\perp}) && \text{by the projection formula,}\\
&= \pi_*(\delta^{n-k-1} \cap 1_{\mathrm{M},\mathrm{M}^\perp})(\sigma_{\mathcal{F}}).
\end{aligned}$$

The result then follows by the definition of the Chern–Schwartz-MacPherson cycle of M. □

The following property of the Chern–Schwartz–MacPherson cycles of matroids generalizes [Alu13, Theorem 1.2].

Proposition 5.1 ([LdMRS20], Thm. 5.8)**.** For each $0 \leqslant k \leqslant r$, we have

$$\alpha^k \cap \mathrm{csm}_k(\mathrm{M}) = (-1)^{r-k} h_{r-k}(\mathrm{BC}(\mathrm{M}))$$

Theorem 1.2. When M has no loops and no coloops, we have

$$\overline{\chi}_{\mathrm{M}}(q+1) = \sum_{k=0}^{r} (-1)^{r-k} \deg(\gamma^k\, \delta^{n-k-1}) q^k.$$

Proof. For each $0 \leqslant k \leqslant r$,

$$\begin{aligned}
h_{r-k}(\mathrm{BC}(\mathrm{M})) &= (-1)^{r-k}\alpha^k \cap \mathrm{csm}_k(\mathrm{M}) && \text{by Proposition 5.1,}\\
&= \alpha^k \cap \pi_*(\delta^{n-k-1} \cap 1_{\mathrm{M},\mathrm{M}^\perp}) && \text{by Theorem 1.1,}\\
&= \pi_*\big(\pi^*\alpha^k \cap (\delta^{n-k-1} \cap 1_{\mathrm{M},\mathrm{M}^\perp})\big) && \text{by the projection formula,}\\
&= \pi_*\big(\gamma^k\delta^{n-k-1} \cap 1_{\mathrm{M},\mathrm{M}^\perp}\big),
\end{aligned}$$

as desired. □

6. Tropical Hodge theory

6.1. **Lefschetz fans.** For a simplicial fan Σ in a vector space N, we continue to let $\mathcal{K}(\Sigma) \subseteq A^1(\Sigma)$ denote the cone of strictly convex piecewise linear functions on Σ. We recall that the Lefschetz property for Σ (Definition 1.5) involves five conditions. A Lefschetz fan has (1) a fundamental weight $w \in \mathrm{MW}_d(\Sigma)$ which induces Poincaré duality (2). We shall abbreviate the latter by PD. The Hard Lefschetz property (3) and Hodge–Riemann relations (4) are statements that hold for all $0 \leqslant k \leqslant d/2$ and all $\ell \in \mathcal{K}(\Sigma)$: we will call those statements $\mathrm{HL}^k(\ell)$ or $\mathrm{HR}^k(\ell)$, respectively, and say that Σ satisfies HL^k if $\mathrm{HL}^k(\ell)$ is true for all $\ell \in \mathcal{K}(\Sigma)$, and that Σ satisfies HL if it satisfies HL^k for all k. We will use HR^k and $\mathrm{HR}^k(\ell)$ analogously. If $\mathcal{K}(\Sigma)$ is empty, then of course the

HL and HR properties hold vacuously. The hereditary property (5) says that stars of cones in Lefschetz fans are also Lefschetz.

Definition 6.1 (Mixed Lefschetz)**.** We say that

(3′) Σ has the *mixed* Hard Lefschetz property if, for all $0 \leqslant k \leqslant d/2$ and $\ell_1, \ldots, \ell_{d-2k} \in \mathcal{K}(\Sigma)$, the multiplication map
$$L\cdot : A^k(\Sigma) \to A^{d-k}(\Sigma)$$
is an isomorphism, where $L := \ell_1 \cdots \ell_{d-2k}$, and

(4′) Σ satisfies the *mixed* Hodge–Riemann relations if, for all $0 \leqslant k \leqslant d/2$, all $\ell \in \mathcal{K}(\Sigma)$ and all L as above, the bilinear form on $A^k(\Sigma)$ defined by
$$\langle u_1, u_2 \rangle_L := (-1)^k \deg(L \cdot u_1 u_2)$$
is positive-definite when restricted to the subspace $PA^k(\Sigma, L, \ell) := \ker(\ell L\, \cdot)$.

Clearly the mixed properties imply the ordinary ones. Cattani showed that the converse is true as well in [Cat08] using the results from [CKS87]. Since the mixed HR property is particularly convenient for applications such as Theorem 1.4, we include a self-contained proof that Lefschetz fans also possess the "mixed" properties $(3')$ and $(4')$; see Theorem 6.20.

Example 6.2. If Σ is a complete, unimodular, polyhedral simplicial fan, then Σ is Lefschetz. In this case, $A^k(\Sigma) \cong H^{2k}(X_\Sigma, \mathbb{R})$ for all k, where X_Σ is the normal projective toric variety constructed from the fan Σ. Here, the Lefschetz properties follow because X_Σ is a smooth projective variety, and $\mathcal{K}(\Sigma)$ is the cone of Kähler forms on X_Σ.

6.2. **The ample cone.** Let us look at the cone $\mathcal{K}(\Sigma)$ in more detail. We say a piecewise linear function $\phi\colon \Sigma \to \mathbb{R}$ is positive on Σ if $\phi(x) > 0$ for all non-zero $x \in |\Sigma|$, and say an equivalence class $\ell \in A^1(\Sigma)$ is positive if it has a positive representative. The *(open) effective cone*, is defined to be the set $\mathrm{Eff}^\circ(\Sigma) \subseteq A^1(\Sigma)$ of positive classes.

For each cone σ of Σ, the subfan $\overline{\mathrm{st}}_\Sigma(\sigma) \subseteq \Sigma$ maps to the star $\mathrm{st}_\Sigma(\sigma)$ under the linear projection $N \to N/\operatorname{span}(\sigma)$. This map is a Chow equivalence, so we will identify the Chow rings of $\overline{\mathrm{st}}_\Sigma(\sigma)$ and $\mathrm{st}_\Sigma(\sigma)$, and let $\iota_\sigma^*\colon A(\Sigma) \to A(\mathrm{st}_\Sigma(\sigma))$ denote pullback along the inclusion.

Definition 6.3. If Σ is a Lefschetz fan, the Kähler (or ample) cone of Σ is defined recursively: if Σ is 1-dimensional, then $\mathcal{K}(\Sigma) = \mathrm{Eff}^\circ(\Sigma)$. Otherwise,
$$\mathcal{K}(\Sigma) := \left\{\ell \in A^1(\Sigma) \colon \ell \in \mathrm{Eff}^\circ(\Sigma) \text{ and } \iota_\sigma^*(\ell) \in \mathcal{K}(\mathrm{st}_\Sigma(\sigma)) \text{ for all } \sigma \in \Sigma\right\}.$$

Clearly, $\ell \in \mathcal{K}(\Sigma)$ if and only if $\iota_\sigma^*(\ell) \in \mathrm{Eff}^\circ(\mathrm{st}_\Sigma(\sigma))$ for all $\sigma \in \Sigma$. Geometrically, this means that ℓ is in the Kähler cone if and only if, for each cone σ, ℓ has a piecewise linear representative ϕ supported on $\overline{\mathrm{st}}_\Sigma(\sigma)$ which is zero on σ and positive on the cones containing σ. That is, ℓ is the class of a piecewise linear function which is strictly convex around each σ.

Proposition 6.4. The set $\mathcal{K}(\Sigma)$ is an open polyhedral cone, and $\iota_\sigma^* \mathcal{K}(\Sigma) \subseteq \mathcal{K}(\mathrm{st}_\Sigma(\sigma))$ for all $\sigma \in \Sigma$.

Proof. The property of being a polyhedral cone is preserved under finite intersections. The second claim follows from the definition. □

A fan Σ is *quasiprojective* if it is a subfan of the normal fan of a (strictly convex) polytope. If Σ is quasiprojective, the cone $\mathcal{K}(\Sigma)$ is nonempty.

If we replace strict inequalities with weak ones above, we arrive instead at the nef cone $\mathcal{L}_\Sigma$ defined in [GM12]. This is a nonempty, closed polyhedral cone in $A^1(\Sigma)$. If $\mathcal{L}_\Sigma$ is full-dimensional, then $\mathcal{K}(\Sigma)$ is the interior of $\mathcal{L}_\Sigma$. Otherwise, $\mathcal{K}(\Sigma)$ is empty.

6.3. **Stellar subdivisions.** Now we focus on the effect of a single blowup of a toric variety along a torus orbit closure. On the level of fans, this is realized by a stellar subdivision. More precisely, suppose Σ is simplicial, $\sigma \in \Sigma$ is a cone, and $V(\sigma)$ denotes the corresponding closed orbit. We recall (see [CLS11, §3.3]) that $\mathrm{Bl}_{V(\sigma)}(X_\Sigma) = X_{\widetilde{\Sigma}}$, where the fan $\widetilde{\Sigma} \coloneqq \mathrm{stellar}_\sigma \Sigma$. Let ρ be the unique element of $\widetilde{\Sigma}(1) - \Sigma(1)$: then

$$\mathbf{e}_\rho = \sum_{\eta \in \sigma(1)} \mathbf{e}_\eta, \tag{6.3.1}$$

where $\mathbf{e}_\nu$ denotes a primitive vector generating the ray ν. Let $p\colon \widetilde{\Sigma} \to \Sigma$ be the map of fans induced by the identity map on N.

Definition 6.5. At this point, we distinguish two possibilities. In the first, every closed orbit in X_Σ meets $V(\sigma)$. In terms of fans, this means $\Sigma = \overline{\mathrm{st}}_\Sigma(\sigma)$, and $\widetilde{\Sigma} = \overline{\mathrm{st}}_{\widetilde{\Sigma}}(\rho)$. In this case, $A(\Sigma) \cong A(\mathrm{st}_\Sigma(\sigma))$ and $A(\widetilde{\Sigma}) \cong A(\mathrm{st}_{\widetilde{\Sigma}}(\rho))$, which are Chow rings of fans of dimensions $\dim(\Sigma) - d$ and $\dim(\Sigma) - 1$, respectively, where $d = \dim(\sigma)$. We will call this a *star-shaped* subdivision (obtained by blowing up a star.) $\widetilde{\Sigma}$. Otherwise, we will say the stellar subdivision is *ordinary*.

The star-shaped subdivision has an alternative interpretation. The stars $\mathrm{st}_\Sigma(\sigma)$ and $\mathrm{st}_{\widetilde{\Sigma}}(\rho)$ are fans in $N/\langle\sigma\rangle$ and $N/\langle\rho\rangle$, respectively. The quotient map $N/\langle\rho\rangle \to N/\langle\sigma\rangle$ induces a map of fans $\mathrm{st}_{\widetilde{\Sigma}}(\rho) \to \mathrm{st}_\Sigma(\sigma)$. The corresponding map of toric varieties is a $\mathbb{P}^{d-1}$-bundle. We refer to [CLS11, §3.3] for details. For trivial reasons, Σ and $\widetilde{\Sigma}$ cannot be Lefschetz; however, $\mathrm{st}_{\widetilde{\Sigma}}(\rho)$ and $\mathrm{st}_\Sigma(\sigma)$ may be.

Now we relate the Chow rings of Σ and $\widetilde{\Sigma}$. We continue to let ρ denote the ray that subdivides the cone σ. Recall that $p^*\colon A(\Sigma) \to A(\widetilde{\Sigma})$ gives $A(\widetilde{\Sigma})$ the structure of a $A(\Sigma)$-module. Since p is a proper map of fans (see [CLS11, Thm. 3.4.11]), there is a Gysin pushforward map $p_*\colon A(\widetilde{\Sigma}) \to A(\Sigma)$ which is a homomorphism of $A(\Sigma)$-modules. A special case of Brion's formula [Bri96, Thm. 2.3] states that, for each cone $\widetilde{\tau}$,

$$p_*(x_{\widetilde{\tau}}) = \begin{cases} x_\tau & \text{if } p(\widetilde{\tau}) \subseteq \tau \text{ and } \dim\widetilde{\tau} = \dim\tau; \\ 0 & \text{otherwise.} \end{cases} \tag{6.3.2}$$

Lemma 6.6. The pullback homomorphism $p^*\colon A(\Sigma) \to A(\widetilde{\Sigma})$ is defined in degree 1 by the formula

$$x_\nu \mapsto \begin{cases} x_\nu & \text{if } \nu \notin \sigma(1); \\ x_\nu + x_\rho & \text{if } \nu \in \sigma(1). \end{cases} \tag{6.3.3}$$

Proof. Since $A(\Sigma)$ is generated in degree 1, it is sufficient to check that p induces the map of piecewise linear functions in (6.3.3). To avoid confusion, we temporarily denote the Courant functions on $\widetilde{\Sigma}$ by $\left\{\widetilde{x}_\nu \colon \nu \in \widetilde{\Sigma}(1)\right\}$.

Consider the function $p^* x_\nu$ on the fan $\widetilde{\Sigma}$. For rays ν not in σ, clearly $p^* x_\nu = \widetilde{x}_\nu$. For $\nu \in \sigma(1)$, we check that the functions $p^* x_\nu$ and $\widetilde{x}_\nu + \widetilde{x}_\rho$ agree on each ray $\mu \in \widetilde{\Sigma}(1)$: since they are piecewise linear, this implies they are equal. Indeed, for $\mu \neq \rho$, we have $p^* x_\nu(\mathbf{e}_\mu) = \widetilde{x}_\nu(\mathbf{e}_\mu) = \delta_{\mu,\nu}$ and $\widetilde{x}_\rho(\mu) = 0$. For $\mu = \rho$,

$$\begin{aligned} \widetilde{x}_\nu(\mathbf{e}_\rho) + \widetilde{x}_\rho(\mathbf{e}_\rho) &= 0 + 1 \\ &= x_\nu(\mathbf{e}_\rho), \end{aligned}$$

because x_ν is linear on σ and the coefficient of $\mathbf{e}_\nu$ in $\mathbf{e}_\rho$ equals 1, by (6.3.1). ☐

Proposition 6.7. If $\widetilde{\Sigma}$ is a stellar subdivision of Σ, then $p^*\colon A^k(\Sigma) \to A^k(\widetilde{\Sigma})$ is injective for all k, and an isomorphism for $k = d$.

Proof. Using the formula (6.3.2) for pushforward and Lemma 6.6 for pullback, we see $p_* p^*$ is the identity function on $A^0(\Sigma)$. Now $A(\Sigma)$ is generated by 1 as an $A(\Sigma)$-module, so $p_* p^* = 1$ in all degrees.

It follows that p^* is injective. To check that it is also surjective in top degree, we check the dual statement instead, that

$$p_*\colon \mathrm{MW}_d(\widetilde{\Sigma}) \to \mathrm{MW}_d(\Sigma)$$

is injective. For this, let $w \in \mathrm{MW}_d(\Sigma)$ be a non-zero Minkowski weight on the maximal cones of $\widetilde{\Sigma}$. For $\sigma \in \Sigma(d)$, we have

$$p_*(w)(\sigma) = w(\widetilde{\sigma}),$$

provided that $p(\widetilde{\sigma}) \subseteq \sigma$. Clearly $p_*(w) \neq 0$, so p_* is injective. ☐

Our goal in the next few pages is to understand how the Lefschetz property behaves under edge subdivisions, so our first step is the Chow ring.

Theorem 6.8. Let $p\colon \widetilde{\Sigma} \to \Sigma$ be the map of fans given by subdividing an edge $\sigma \in \Sigma(2)$ with a ray $\rho \in \widetilde{\Sigma}(1)$. There is an isomorphism of graded $A(\Sigma)$-modules

$$A^i(\widetilde{\Sigma}) \cong A^i(\Sigma) \oplus x_\rho \cdot A^{i-1}(\mathrm{st}_\Sigma(\sigma)), \tag{6.3.4}$$

for all $i \geqslant 0$.

To prove Theorem 6.8, we consider the subdivision of Σ, restricted to the star of σ. It is not hard to see that $p\colon \widetilde{\Sigma} \to \Sigma$ restricts to a star-shaped subdivision: the star of σ within Σ:

$$\begin{tikzcd} \overline{\mathrm{st}}_{\widetilde{\Sigma}}(\rho) \arrow[r, hook, "j"] \arrow[d, "p_\sigma"] & \widetilde{\Sigma} \arrow[d, "p"] \\ \overline{\mathrm{st}}_{\Sigma}(\sigma) \arrow[r, hook, "i_\sigma"] & \Sigma \end{tikzcd} \tag{6.3.5}$$

Keel [Kee92, Thm. 1 (Appendix)] relates the Chow rings of the star-shaped subdivision.

Lemma 6.9. For any cone $\sigma \in \Sigma(k)$ with $k \geqslant 2$, let $\widetilde{\Sigma} = \mathrm{stellar}_\sigma(\Sigma)$. Then there is an algebra isomorphism

$$A(\mathrm{st}_{\widetilde{\Sigma}}(\rho)) \cong A(\mathrm{st}_\Sigma(\sigma))[t]/(t^k + c_1 t^{k-1} + \cdots + c_k)$$

induced by the pullback p_σ^*, where $c_i \in A^i(\mathrm{st}_\Sigma(\sigma))$ are the Chern classes of the normal bundle of $V(\sigma)$ in $X_{\mathrm{st}_\Sigma(\sigma)}$, for $1 \leqslant i \leqslant k$. Under the isomorphism, $x_\rho \mapsto -t$.

Proof of Theorem 6.8. Let $p\colon \widetilde{\Sigma} \to \Sigma$ be an edge subdivision, and apply the Chow functor to the square (6.3.5). The maps i_σ^* and j^* are surjective, since they are clearly surjective in degree 1. The vertical maps are injective, by Proposition 6.7. We obtain short exact sequences of $A(\Sigma)$-modules, where J_i and C_i for $i = 1, 2$ denote the respective kernels and cokernels:

$$\begin{tikzcd} & C_2 & C_1 \arrow[l, "\cong"'] & & \\ 0 & A(\mathrm{st}_{\widetilde{\Sigma}}(\rho)) \arrow[l] \arrow[u, two heads] & A(\widetilde{\Sigma}) \arrow[l, "j^*"'] \arrow[u, two heads] \arrow[d, dotted, "p_*"] & J_2 \arrow[l] & 0 \arrow[l] \\ 0 & A(\mathrm{st}_\Sigma(\sigma)) \arrow[l] \arrow[u, hook, "p_\sigma^*"] & A(\Sigma) \arrow[l, "i_\sigma^*"] \arrow[u, hook, "p^*"] & J_1 \arrow[l] \arrow[u, "\cong"] & 0 \arrow[l] \end{tikzcd}$$

From [Kee92], it follows that $J_1 \cong J_2$, so $C_1 \cong C_2$ by the Snake Lemma. Lemma 6.9 says $C_2 \cong x_\rho \cdot A(\mathrm{st}_\Sigma(\sigma))$. To see that p^* is a split injection, we recall that p_* is a left inverse to p^*. $\square$

Remark 6.10. We have used a pushforward of Minkowski weights for any map $f\colon \Sigma \to \Sigma'$ of fans, and a Gysin pushforward of Chow groups which is defined only for proper maps. If Σ and Σ' have the same dimension, these can be shown to agree. More generally, if Σ and Σ' have Poincaré duality in dimensions d and d', respectively, the pushforward $f_*\colon \mathrm{MW}(\Sigma) \to \mathrm{MW}(\Sigma')$ gives a map $f_*\colon A(\Sigma) \to A(\Sigma')[d' - d]$ via the Poincaré duality isomorphisms

$$\mathrm{MW}_i(\Sigma) \cong A^{d-i}(\Sigma) \text{ and } \mathrm{MW}_i(\Sigma) \cong A^{d'-i}(\Sigma')$$

for all i.

In particular, if σ is a k-dimensional cone in a Lefschetz fan Σ, then by definition both $\mathrm{st}_\Sigma(\sigma)$ and Σ satisfy PD. So we obtain a pushforward map

$$i_*^\sigma\colon A(\mathrm{st}_\Sigma(\sigma)) \to A(\Sigma)[k].$$

It has the property that

$$i_*^\sigma i_\sigma^* \colon A(\Sigma) \to A(\Sigma)[1]$$

is given by multiplication by x_σ.

Finally, we note that the pullback of $\mathcal{K}(\Sigma)$ lies in the boundary of $\mathcal{K}(\widetilde{\Sigma})$ along an edge subdivision.

Lemma 6.11. If $\ell \in p^*(K(\Sigma))$, then $\ell - \epsilon \cdot x_\rho \in K(\widetilde{\Sigma})$ for sufficiently small values of $\epsilon > 0$.

Proof. If τ is a cone of $\widetilde{\Sigma}$ which belongs to Σ, then $\overline{\mathrm{st}}_{\widetilde{\Sigma}}(\tau) \subseteq \overline{\mathrm{st}}_{\Sigma}(\tau)$. So ℓ is the class of a strictly convex function ϕ around τ. Strict convexity is an open condition, so $\phi_\epsilon \coloneqq \phi - \epsilon \cdot x_\rho$ has the same property for ϵ sufficiently close to 0.

Otherwise, τ contains the ray ρ, so τ is not a cone of Σ, and ℓ is the class of a linear function ϕ on the closed star of τ. In that case, ϕ_ϵ agrees with ϕ on the link of τ, and is strictly smaller inside τ, provided $\epsilon > 0$. That is, ϕ_ϵ is strictly convex around τ.

Combining the conditions, $\ell - \epsilon \cdot x_\rho \in K(\widetilde{\Sigma})$ for sufficiently small, positive ϵ. □

6.4. **Signatures of Hodge–Riemann forms.** Suppose that multiplication by some element $L \in A^{d-2k}(\Sigma)$ is an isomorphism in degree k. One can check directly that the real bilinear form $\mathrm{hr}^k(\Sigma, L)$ is nondegenerate, which is to say that it has b_i^+ positive eigenvalues and b_i^- negative eigenvalues, where $b_i^+ + b_i^- = b_i(\Sigma) \coloneqq \dim_{\mathbb{R}} A^i(\Sigma)$. Its signature, $b_i^+ - b_i^-$, can be used to characterize the HR property. This useful fact appears as [AHK18, Prop. 7.6], as well as [McM93, Thm. 8.6] in the case when $L = \ell^{d-2k}$.

Proposition 6.12. Suppose Σ satisfies PD, and $U \subseteq A^{d-2k}(\Sigma)$ is a connected set in the Euclidean topology. For a fixed $k \leqslant d/2$, if $\mathrm{hr}^k(\Sigma, L)$ is nondegenerate on $A^k(\Sigma)$ for all $L \in U$, then the signature of $\mathrm{hr}^k(\Sigma, L)$ is constant for all $L \in U$.

Proof. The eigenvalues of $\mathrm{hr}^k(\Sigma, L)$ are real, and they vary continuously with L. By hypothesis, they are all non-zero for $L \in U$, so their signs (taken as a multiset) are constant on U, because U is connected. □

Theorem 6.13 (The HR signature test)**.** Suppose Σ satisfies PD and $k \leqslant d/2$ is an integer for which

(1) $\mathrm{hr}^i(\Sigma, L)$ is nondegenerate for all $0 \leqslant i \leqslant k$ and all $L \in \mathrm{Sym}^{d-2i}\,\mathcal{K}(\Sigma)$, and

(2) $\mathrm{hr}^i(\Sigma, L)$ is positive-definite on $PA^i(\Sigma, L, \ell_0)$, for all $\ell_0 \in \mathcal{K}(\Sigma)$ and $i < k$.

Then, for any $L \in \mathrm{Sym}^{d-2k}\,\mathcal{K}(\Sigma)$, the form $\mathrm{hr}^k(\Sigma, L)$ is positive-definite on $PA^k(\Sigma, L, \ell_0)$ for all $\ell_0 \in \mathcal{K}(\Sigma)$ if and only if its signature on $A^k(\Sigma)$ equals

$$\sum_{i=0}^{k} (-1)^{k-i}\big(b_i(\Sigma) - b_{i-1}(\Sigma)\big). \tag{6.4.1}$$

Proof. In the special case where $L = \ell_0^{d-2k}$, we refer to [AHK18, Prop. 7.6]. Since $\mathcal{K}(\Sigma)$ is connected, and $U \coloneqq \mathrm{Sym}^{d-2k}\,\mathcal{K}(\Sigma)$ is a quotient of $\mathcal{K}(\Sigma)^{\times(d-2k)}$, the set U is also connected. So a general Lefschetz element has the same signature as ℓ_0^{d-2k}, by Proposition 6.12, so $\mathrm{hr}^k(\Sigma, L)$ is positive-definite on its space of primitives because $\mathrm{hr}^k(\Sigma, \ell_0^{d-2k})$ is. $\square$

We note that, if L passes the signature test above, then $b_i(\Sigma) - b_{i-1}(\Sigma) = \dim PA^k(\Sigma, L, \ell_0)$, for any ℓ_0. Schematically, $A(\Sigma)$ looks like

$$
\begin{array}{ccccc}
 & & PA^2 & + & \cdots \\
 & PA^1 & + & - & \cdots \\
PA^0 & + & - & + & \cdots \\
 & A^0 & A^1 & A^2 &
\end{array}
$$

Corollary 6.14. Let Σ be a fan of dimension d satisfying Poincaré duality. Let $k \leqslant d/2$. Suppose Σ satisfies mixed HR^i for all $i < k$, mixed HL^k, as well as $\mathrm{HR}^k(L')$ for some $L' \in \mathrm{Sym}^{d-2k}\,\mathcal{K}(\Sigma)$. Then Σ satisfies HR^k.

Proof. Let $L \in \mathrm{Sym}^{d-2k}\,\mathcal{K}(\Sigma)$ be any element. By the Hard Lefschetz hypothesis, $\mathrm{hr}^k(\Sigma, L)$ is nondegenerate. By Proposition 6.12, it has the same signature as $\mathrm{hr}^k(\Sigma, L')$. Since we assume Σ satisfies mixed HR^i for $i < k$, Theorem 6.13 shows $\mathrm{HR}^k(L) \Leftrightarrow \mathrm{HR}^k(L')$. $\square$

In the special case of a star-shaped blowup, the signature test simplifies slightly. Let $\Delta = \mathrm{st}_\Sigma(\sigma)$ and $\widetilde{\Delta} = \mathrm{st}_{\widetilde{\Sigma}}(\rho)$, where Δ has dimension d.

Corollary 6.15. An element L of degree $k \leqslant (d+1)/2$ has the HR property for $\widetilde{\Delta}$ if and only if the signature of $\mathrm{hr}^k(\widetilde{\Delta}, L)$ equals $b_k(\Delta) - b_{k-1}(\Delta)$.

Proof. By Theorem 6.8, we have $b_k(\widetilde{\Delta}) = b_k(\Delta) + b_{k-1}(\Delta)$ for all $k \leqslant (d+1)/2$. Substituting into (6.4.1) simplifies as shown. $\square$

6.5. **Lefschetz properties under edge subdivision I.** With these preparations, we now set out to show that the Lefschetz property of a fan is unaffected by codimension-2 blowups and blowdowns. The precise statement and its proof appear in Section 6.7 as Theorems 6.26 and 6.27. Here, we get started with Poincaré duality, and we do so for star-shaped subdivisions first.

Proposition 6.16. Let Σ be a simplicial fan, $\sigma \in \Sigma(2)$, and $\widetilde{\Sigma} \coloneqq \mathrm{stellar}_\sigma(\Sigma)$. Then PD holds for $\mathrm{st}_\Sigma(\sigma)$ if and only if it holds for $\mathrm{st}_{\widetilde{\Sigma}}(\rho)$.

Proof. Let $\Delta = \mathrm{st}_\Sigma(\sigma)$ and $\widetilde{\Delta} = \mathrm{st}_{\widetilde{\Sigma}}(\rho)$. Assume that PD holds for at least one of them, and let $d = \dim \widetilde{\Delta}$. By Theorem 6.8, for all $i \geqslant 0$,

$$A^i(\widetilde{\Delta}) \cong A^i(\Delta) \oplus x_\rho A^{i-1}(\Delta).$$

We see that $A^{d-1}(\Delta) \cong A^d(\widetilde{\Delta})$, so if one of Δ or $\widetilde{\Delta}$ has a fundamental weight, they both do. By inspection, $b_i(\Delta) = b_{d-1-i}(\Delta)$ for all i if and only if $b_i(\widetilde{\Delta}) = b_{d-i}(\widetilde{\Delta})$ for all i. So we may assume both sets of equalities hold.

For any $u \in A^i(\widetilde{\Delta})$ and $v \in A^{d-i}(\widetilde{\Delta})$, we write $u = u_0 + u_1 x_\rho$ and $v = v_0 + v_1 x_\rho$ where u_0, u_1, v_0, v_1 are elements of $A(\Delta)$ of degrees i, $i-1$, $d-i$, and $d-1-i$, respectively. Then $u_0 v_0 \in A^d(\Delta) = 0$, and $x_\rho^2 = c_1 \cdot x_\rho + c_2$ for some $c_1, c_2 \in A(\Delta)$. With respect to the decomposition above, the matrix of the multiplication pairing has the form

$$M^i(\widetilde{\Delta}) = \left(\begin{array}{c:c} 0 & -M^{i-1}(\Delta) \\ \hdashline -M^i(\Delta) & * \end{array}\right), \tag{6.5.1}$$

where $M^i(\Delta)$ denotes the matrix of the pairing $A^i(\Delta) \times A^{d-1-i}(\Delta) \to \mathbb{R}$. Thus if each matrix $M^i(\widetilde{\Delta})$ is invertible, so is each matrix $M^i(\Delta)$, and conversely. If either Δ or $\widetilde{\Delta}$ has PD, then they both do. $\square$

Proposition 6.17. Let Σ be a simplicial fan, $\sigma \in \Sigma(2)$, and $\widetilde{\Sigma} \coloneqq \operatorname{stellar}_\sigma(\Sigma)$. Suppose that PD holds for $\operatorname{st}_\Sigma(\sigma)$. Then PD holds for $\widetilde{\Sigma}$ if and only if it holds for Σ.

Proof. For dimensional reasons, if either fan has Poincaré duality, $\widetilde{\Sigma}$ is an ordinary subdivision. Let $d \coloneqq \dim \Sigma = \dim \widetilde{\Sigma}$. By Proposition 6.7, we have $A^d(\widetilde{\Sigma}) \cong A^d(\Sigma)$, and they have a common degree map.

Using the decomposition (6.3.4) and Poincaré duality in $\operatorname{st}_\Sigma(\sigma)$, we have $b_i(\Sigma) = b_{d-i}(\Sigma)$ and $b_i(\widetilde{\Sigma}) = b_{d-i}(\widetilde{\Sigma})$ for all $0 \leqslant i \leqslant d$. Since $A^s(\Sigma) \times A^t(\operatorname{st}_\Sigma(\sigma)) \to A^{s+t}(\operatorname{st}_\Sigma(\sigma))$ is the zero map when $s + t > d - 2$, ordering bases compatibly with (6.3.4) gives a block-diagonal matrix:

$$M^i(\widetilde{\Sigma}) = \left(\begin{array}{c:c} M^i(\Sigma) & 0 \\ \hdashline 0 & M^i(\operatorname{st}_\Sigma(\sigma)) \end{array}\right)$$

Clearly $M^i(\widetilde{\Sigma})$ has full rank if and only if $M^i(\Sigma)$ and $M^{i-1}(\operatorname{st}_\Sigma(\sigma))$ both do as well, which completes the proof. $\square$

Lemma 6.18. If Σ is a d-dimensional simplicial fan with PD, let $I \subseteq \Sigma(1)$ be a subset for which $\{x_\nu : \nu \in I\}$ spans $A^1(\Sigma)$. Then the map

$$\oplus i_\nu^* \colon A^i(\Sigma) \to \bigoplus_{\nu \in S} A^i(\operatorname{st}_\Sigma(\nu))$$

is injective for all $0 \leqslant i < d$.

Proof. Suppose $i_\nu^*(u) = 0$ for each ray ν. Then $i^\nu_* i_\nu^*(u) = x_\nu \cdot u = 0$ for a set of generators x_ν of $A(\Sigma)$. Since $A(\Sigma)$ is Gorenstein, this implies $u \in A^d(\Sigma)$. $\square$

Proposition 6.19. Let Σ be a simplicial fan satisfying PD in degree d. Suppose that the fan $\operatorname{st}_\Sigma(\nu)$ satisfies mixed HR for each ray $\nu \in \Sigma(1)$. Then Σ satisfies mixed HL.

Proof. Let $L := \ell_1 \cdots \ell_{d-2k}$ be a Lefschetz element, and consider the map $L\cdot : A^k(\Sigma) \to A^{d-k}(\Sigma)$. By PD, we know $b_k(\Sigma) = b_{d-k}(\Sigma)$, so it is enough to show that $L\cdot$ is injective. Suppose, then, that $L \cdot u = 0$ for some $u \in A^k(\Sigma)$.

Let $L' := \ell_2 \cdots \ell_{d-2k}$. For each ray $\nu \in \Sigma(1)$, the pullback $i_\nu^*(L')$ is a Lefschetz element for $\mathrm{st}_\Sigma(\nu)$ by Proposition 6.4. Since $L \cdot u = 0$, each pullback of u is primitive; that is, $i_\nu^*(u) \in PA^k(\mathrm{st}_\Sigma(\nu), i_\nu^*(\ell_1))$.

We may write $\ell_1 = \sum_{\nu\in\Sigma(1)} c_\nu x_\nu$ where each coefficient $c_\nu > 0$, since we can represent ℓ_1 by a PL function which is strictly positive on each ray. Degree commutes with pullback:

$$\begin{aligned}
0 &= \deg_\Sigma(L\cdot u\cdot u)\\
&= \deg_\Sigma\Big(\sum_{\nu\in\Sigma(1)} c_\nu x_\nu L'\cdot u\cdot u\Big)\\
&= \sum_\nu c_\nu \deg_{\mathrm{st}_\Sigma(\nu)}(i_\nu^*(L')\cdot i_\nu^*(u)\cdot i_\nu^*(u))\\
&= (-1)^{k-1}\sum_{\nu\in\Sigma(1)} c_\nu \,\langle i_\nu^*(u), i_\nu^*(u)\rangle_{i_\nu^*(L')}\,.
\end{aligned}$$

Since the c_ν's are strictly positive, each summand is zero, and the mixed HR property in $\mathrm{st}_\Sigma(\nu)$ implies $i_\nu^*(u) = 0$, for each ν. By Lemma 6.18, we have $u = 0$, and $L\cdot$ is injective. □

As an application, we see that the mixed Lefschetz properties in Definition 6.1 are actually no stronger than the pure ones. See [Cat08] for a discussion in a more general context.

Theorem 6.20. If Σ is a Lefschetz fan, then it also has the mixed HL and mixed HR properties.

Proof. We use induction on dimension. If $\dim\Sigma = 1$, the mixed and pure properties are identical, so let us suppose the claim is true for all Lefschetz fans of dimension less than d, for some $d > 1$. Let Σ be a Lefschetz fan of dimension d. By induction, $\mathrm{st}_\Sigma(\nu)$ satisfies mixed HR for all rays $\nu \in \Sigma(1)$. By Proposition 6.19, then Σ satisfies mixed HL.

Now we establish mixed HR for Σ. For any $\ell \in K(\Sigma)$ and $0 \leqslant k \leqslant d/2$, the "pure" property $\mathrm{HR}^k(L')$ holds for $L' = \ell^{d-2k}$. Corollary 6.14 states that mixed HL and mixed HR^i for $i < k$ implies mixed HR^k. Setting $k = 0$, we see Σ has the mixed HR^0 property. Arguing by induction on k, we obtain mixed HR^k for all $k \leqslant d/2$. □

6.6. **Lefschetz properties under edge subdivision II.** Now we examine how the Hodge–Riemann forms fare under stellar subdivisions. We begin with a technical lemma, then the case of an ordinary subdivision.

Lemma 6.21. Suppose $p\colon \widetilde{\Sigma} \to \Sigma$ is an edge subdivision. Then $p_*(x_\rho) = 0$, and $p_*(x_\rho^2) = -x_\sigma$.

Proof. The first claim follows from the pushforward formula (6.3.2). Now let x_1, x_2 be the Courant functions for the rays ν_1, ν_2 of the cone $\sigma \in \Sigma(2)$, so $x_\sigma = x_1 x_2$. By Lemma 6.6, we compute for $i = 1, 2$ that

$$0 = p_*(x_\rho)x_i = p_*\big(x_\rho(x_i + x_\rho)\big),$$

so $p_*(x_\rho x_i) = -p_*(x_\rho^2)$. Since $\{\nu_1, \nu_2\}$ is not contained in a cone of $\widetilde{\Sigma}$, we have

$$x_\sigma = p_* p^*(x_1 x_2) = p_*\big((x_1 + x_\rho)(x_2 + x_\rho)\big) = (0 - 2 + 1)p_*(x_\rho^2). \qquad \square$$

Lemma 6.22. Let Σ be a d-dimensional fan with PD. Suppose $\sigma \in \Sigma(2)$ and $\widetilde{\Sigma} = \operatorname{stellar}_\sigma(\Sigma)$ is an ordinary subdivision. Then, for all $0 \leqslant k \leqslant d/2$ and all $L \in \operatorname{Sym}^{d-2k} \mathcal{K}(\Sigma)$, we have $\operatorname{hr}^k(\widetilde{\Sigma}, p^*L) = \operatorname{hr}^k(\Sigma, L) \oplus \operatorname{hr}^{k-1}(\operatorname{st}_\Sigma(\sigma), i_\sigma^*(L))$, an orthogonal direct sum.

Proof. We consider $\operatorname{hr}^k(\widetilde{\Sigma}, p^*L)$ under the direct sum decomposition (6.3.4). Given elements $(a, 0)$ and $(0, b) \in A^k(\Sigma) \oplus A^{k-1}(\operatorname{st}_\Sigma(\sigma))$, we calculate as follows.

Since i_σ^* is surjective, we may write $b = i_\sigma^*(b')$ for some $b' \in A(\Sigma)$. Then

$$\begin{aligned}
(-1)^k \langle (a,0), (0,b) \rangle &= \deg_{\widetilde{\Sigma}} \big(p^*(L) \cdot p^*(a) \cdot j_* p_\sigma^*(b)\big) \\
&= \deg_{\widetilde{\Sigma}} \big(p^*(L) p^*(a) \cdot p^* i^*(b) \cdot x_\rho\big) \\
&= \deg_\Sigma \big(L \cdot ab' \cdot p_*(x_\rho)\big) \\
&= 0,
\end{aligned}$$

because $p_*(x_\rho) = 0$.

If $a, b \in A^k(\Sigma)$, the equality $\langle (a,0), (b,0) \rangle_{p^*(L)} = \langle a, b \rangle_L$ is straightforward. If $a, b \in A^{k-1}(\operatorname{st}_\Sigma(\sigma))$, again write $a = \iota_\sigma^*(a')$ and $b = \iota_\sigma^*(b')$ for some $a', b' \in A^{k-1}(\Sigma)$. Then, calculating as above,

$$\begin{aligned}
\langle (a,0), (0,b) \rangle &= (-1)^k \deg_{\widetilde{\Sigma}} \big(p^*(L) \cdot p^*(a') p^*(b') \cdot x_\rho^2\big) \\
&= (-1)^k \deg_\Sigma \big(L \cdot a'b' \cdot p_*(x_\rho^2)\big) \\
&= -(-1)^k \deg_\Sigma \big(L \cdot a'b' \cdot x_\sigma\big) \\
&= (-1)^{k-1} \deg_{\operatorname{st}_\Sigma(\sigma)} \big(i_\sigma^*(L) \cdot i_\sigma^*(a') i_\sigma^*(b')\big) \\
&= \langle a, b \rangle_{i_\sigma^*(L)}
\end{aligned}$$

The result follows. $\square$

Next we address star-shaped subdivisions. Let $d = \dim \operatorname{st}_\Sigma(\sigma) = \dim \Sigma - 2$.

Lemma 6.23. Suppose P and Q are $n \times n$ matrices with real entries and $Q = Q^T$. Let

$$M := \begin{pmatrix} 0 & P \\ P^T & Q \end{pmatrix}.$$

If P is nonsingular, then M has signature zero.

Proof. Assume first that Q is invertible, and let $S = -PQ^{-1}P^T$ (the Schur complement.) Then it is easily seen that M is congruent to a block-diagonal matrix:

$$M = \begin{pmatrix} I_n & PQ^{-1} \\ 0 & I_n \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} I_n & 0 \\ Q^{-1}P^T & I_n \end{pmatrix},$$

and the signature of S is the negative of the signature of Q. It follows that M has signature zero.

Now suppose Q is singular. We replace Q by $Q(\epsilon)$ to define $M(\epsilon)$ as above, for some real, invertible symmetric matrices $Q(\epsilon)$ with $\lim_{\epsilon \to 0} Q(\epsilon) = Q$. Then $\det(M(\epsilon)) = (-1)^n \det(P)^2 \neq 0$, regardless of ϵ, so the argument above shows $M(\epsilon)$ has n positive eigenvalues and n negative eigenvalues. By continuity, so does M. □

The last result in this section relates HL and HR along an edge subdivision.

Proposition 6.24. Suppose that at least one of $\mathrm{st}_\Sigma(\sigma)$ and $\mathrm{st}_{\widetilde{\Sigma}}(\rho)$ has Poincaré duality, and that $\ell \in \mathcal{K}(\mathrm{st}_\Sigma(\sigma))$ has the Hard Lefschetz property. Then

- $\ell_\epsilon \coloneqq \ell - \epsilon \cdot x_\rho \in \mathcal{K}(\mathrm{st}_{\widetilde{\Sigma}}(\rho))$ has the HL property for sufficiently small $\epsilon > 0$, and
- For such ϵ, the fan $\mathrm{st}_{\widetilde{\Sigma}}(\rho)$ satisfies $\mathrm{HR}(\ell_\epsilon)$ if $\mathrm{st}_\Sigma(\sigma)$ satisfies $\mathrm{HR}(\ell)$.

Proof. Let $\Delta = \mathrm{st}_\Sigma(\sigma)$ and $\widetilde{\Delta} = \mathrm{st}_{\widetilde{\Sigma}}(\rho)$. By Proposition 6.17, we may assume both Δ and $\widetilde{\Delta}$ have Poincaré duality. By Lemma 6.11, we have $\ell_\epsilon \in K(\widetilde{\Delta})$ for small enough positive ϵ.

If $k < (d+1)/2$, we use the HR property of $\ell \in K(\Delta)$ and (6.3.4) to obtain a decomposition

$$A^k(\widetilde{\Delta}) = PA^k(\Delta, \ell) \oplus \ell A^{k-1}(\Delta) \oplus x_\rho A^{k-1}(\Delta),$$

with respect to which $\mathrm{hr}^k(\widetilde{\Delta}, \ell_\epsilon)$ is represented by a block matrix

$$\mathrm{hr}^k(\Delta, \ell_\epsilon) = \begin{pmatrix} H_{11}(\epsilon) & H_{12}(\epsilon) & H_{13}(\epsilon) \\ H_{21}(\epsilon) & H_{22}(\epsilon) & H_{23}(\epsilon) \\ H_{31}(\epsilon) & H_{32}(\epsilon) & H_{33}(\epsilon) \end{pmatrix}.$$

For any $\epsilon > 0$, the matrix above is congruent to the matrix

$$\overline{\mathrm{hr}}^k(\epsilon) \coloneqq \begin{pmatrix} \epsilon^{-1}H_{11}(\epsilon) & \epsilon^{-1}H_{12}(\epsilon) & H_{13}(\epsilon) \\ \epsilon^{-1}H_{21}(\epsilon) & \epsilon^{-1}H_{22}(\epsilon) & H_{23}(\epsilon) \\ H_{31}(\epsilon) & H_{32}(\epsilon) & \epsilon H_{33}(\epsilon) \end{pmatrix}, \tag{6.6.1}$$

the entries of which we will see are polynomial in ϵ. For elements $p_1, p_2 \in PA^k(\Delta, \ell)$, we have

$$\begin{aligned} \langle p_1, p_2 \rangle_{\ell_\epsilon} &= (-1)^k \deg_{\widetilde{\Delta}} \left((\ell - \epsilon x_\rho)^{d+1-2k} p_1 p_2 \right) \\ &= -(-1)^k \cdot \epsilon \cdot \deg_{\widetilde{\Delta}} \left(\ell^{d-2k} (d+1-2k) p_1 p_2 x_\rho \right) + O(\epsilon^2) \\ &= (-1)^k \epsilon (d+1-2k) \deg_\Delta \left(\ell^{d-2k} p_1 p_2 \right) + O(\epsilon^2) \\ &= (d+1-2k)\epsilon \cdot \langle p_1, p_2 \rangle_\ell + O(\epsilon^2). \end{aligned}$$

so the block $H_{11}(\epsilon)$ represents a positive multiple of the pairing $\mathrm{hr}^k(\Delta, \ell)$, modulo ϵ^2.

Similar computations show that the block $H_{22}(\epsilon)$ is the matrix of the pairing $(d+1-2k)\epsilon \cdot \mathrm{hr}^{k-1}(\Delta,\ell)$, modulo ϵ^2, and $H_{23}(\epsilon) = H_{32}(\epsilon) = -\mathrm{hr}^{k-1}(\Delta,\ell)$ modulo ϵ. Along the same lines, we see $H_{12}(\epsilon) = H_{21}(\epsilon)$ are divisible by ϵ^2, and $H_{13}(\epsilon) = H_{31}(\epsilon)$ is divisible by ϵ. Returning to (6.6.1), we have

$$\overline{\mathrm{hr}}^k(\epsilon) = \begin{pmatrix} (d+1-2k)\,\mathrm{hr}^k(\Delta,\ell)\,|_{PA^k} & 0 & 0 \\ 0 & -(d+1-2k)\,\mathrm{hr}^{k-1}(\Delta,\ell) & -\mathrm{hr}^{k-1}(\Delta,\ell) \\ 0 & -\mathrm{hr}^{k-1}(\Delta,\ell) & 0 \end{pmatrix} + O(\epsilon).$$

Given our assumption that $k < (d+1)/2$, the matrix $\overline{\mathrm{hr}}^k(0)$ is invertible, because each nonzero block is nondegenerate (since ℓ has the HL property). It follows that ℓ_ϵ has the HL^k property for all $0 \leqslant k < (d+1)/2$, for some sufficiently small $\epsilon > 0$. Using Lemma 6.23, we see the signature of $\overline{\mathrm{hr}}^k(\epsilon)$ agrees with that of the top-left block. By hypothesis, $\mathrm{hr}^k(\Delta,\ell)$ is positive-definite on $PA^k(\Delta,\ell)$. Now $\dim PA^k(\Delta,\ell) = b_k(\Delta) - b_{k-1}(\Delta)$, which by Corollary 6.15 is the expected signature for $\overline{\mathrm{hr}}^k(\epsilon)$; that is, HR$^k(\ell_\epsilon)$ holds for sufficiently small ϵ.

It remains to consider the case where d is odd and $k = (d+1)/2$. In this case we have $A^k(\widetilde{\Delta}) = A^{k-1}(\Delta) \oplus x_\rho A^{k-1}(\Delta)$, and (up to a sign) the pairing is just the Poincaré pairing $M^k(\widetilde{\Delta})$. In the middle dimension, $M^k(\Delta) = M^{k-1}(\Delta)$, so we have a block decomposition from (6.5.1):

$$M^k(\widetilde{\Delta}) = \begin{pmatrix} 0 & -M^k(\Delta) \\ -M^k(\Delta) & Q \end{pmatrix}$$

for some square matrix Q. The matrix $M^k(\Delta)$ is nonsingular, by HL^k, so $M^k(\widetilde{\Delta})$ has signature zero by Lemma 6.23, which shows ℓ_ϵ has HR^k for any ϵ by Corollary 6.15 again. □

6.7. **Proofs of the main results.** We are now ready to prove the main result of this section. We treat the star-shaped and ordinary cases separately, beginning with the former.

We will need to use a result of Włodarczyk [Wło97, Theorem A]:

Theorem 6.25. If Σ and Σ' are two smooth, simplicial fans and $|\Sigma| = |\Sigma'|$, there exists a sequence of simplicial fans $\Sigma_0, \Sigma_1, \ldots, \Sigma_N$ for which $\Sigma = \Sigma_0$, $\Sigma_N = \Sigma'$, and Σ_i is obtained from Σ_{i-1} by an edge subdivision or an inverse edge subdivision, for all $1 \leqslant i \leqslant N$.

Proof. By [Wło97, Theorem A], there is a sequence of simplicial fans as above, where either Σ_i is a stellar subdivision of Σ_{i-1}, or vice-versa.

If we regard Σ and Σ' as cones over geometric simplicial complexes, then stellar subdivisions correspond to barycentric subdivisions. Alexander proved [Ale30, Corollary 10:2c] that we may refine the chain of fans above in such a way that each step is the subdivision of an edge, which is to say a cone of codimension 2. □

Theorem 6.26. Let $\widetilde{\Sigma} = \mathrm{stellar}_\sigma(\Sigma)$ be a star-shaped subdivision of a simplicial fan Σ, for some cone $\sigma \in \Sigma(2)$. Then $\mathrm{st}_\Sigma(\sigma)$ is a Lefschetz fan if and only if $\mathrm{st}_{\widetilde{\Sigma}}(\rho)$ is a Lefschetz fan, where ρ is the ray subdividing σ.

Proof. Let $\Delta = \mathrm{st}_{\Sigma}(\sigma)$ and $\widetilde{\Delta} = \mathrm{st}_{\widetilde{\Sigma}}(\rho)$. First, suppose Δ is Lefschetz, and let ν_1, ν_2 denote the two extreme rays of σ. First, we check that the star of each cone $\tau \in \widetilde{\Delta}$ is Lefschetz. This is easy if τ does not contain ν_1 or ν_2, since then τ is a cone of Δ. Otherwise, τ contains (exactly) one such ray, say ν_1. The remaining rays of τ span a cone τ' of Δ, and by inspection, $\mathrm{st}_{\widetilde{\Delta}}(\tau) = \mathrm{st}_{\Delta}(\tau')$, which is again Lefschetz by hypothesis.

The PD property for $\widetilde{\Delta}$ follows from Proposition 6.17. To establish HL, we use Proposition 6.19. For this, we need to know that the star of each ray satisfies mixed HR, but the star of a ray of $\widetilde{\Delta}$ is also a star in Δ, so HL for $\widetilde{\Delta}$ follows. Finally, we use Proposition 6.24: for any $\ell \in \mathcal{K}(\Delta)$, there exists some $\ell_\epsilon \in \mathcal{K}(\widetilde{\Delta})$ with the HR property. By Corollary 6.14, $\widetilde{\Delta}$ has HR.

The converse is trivial: if $\widetilde{\Delta}$ is Lefschetz, then $\mathrm{st}_{\widetilde{\Delta}}(\nu_1) = \Delta$, so Δ is Lefschetz too. $\square$

We note that, in this case, $\mathcal{K}(\Delta)$ is nonempty if and only if $\mathcal{K}(\widetilde{\Delta})$ is nonempty. The forward implication follows immediately from Lemma 6.11. The converse holds by Proposition 6.4, since Δ is a star in $\widetilde{\Delta}$. For arbitrary subdivisions, however, $\mathcal{K}(\widetilde{\Sigma})$ can be nonempty while $\mathcal{K}(\Sigma)$ is empty.

Theorem 6.27. Let Σ be a simplicial fan and $\sigma \in \Sigma(2)$. If Σ is a Lefschetz fan and $K(\Sigma) \neq \varnothing$, then $\widetilde{\Sigma} \coloneqq \mathrm{stellar}_\sigma(\Sigma)$ is Lefschetz. Conversely, if $\widetilde{\Sigma}$ is a Lefschetz fan, then Σ is Lefschetz.

Proof of "$\Rightarrow$": Let $d = \dim \Sigma$: we argue by induction on d. The statement is vacuously true if $d = 1$, so let us assume it holds for all Lefschetz fans of dimension less than d.

First we check that the star of every cone $\tau \in \widetilde{\Sigma}$ is Lefschetz, for which we consider two cases. First suppose $\tau \in \Sigma$. If $\sigma \notin \mathrm{st}_\Sigma(\tau)$, then $\mathrm{st}_{\widetilde{\Sigma}}(\tau) = \mathrm{st}_\Sigma(\tau)$, which is Lefschetz. If, on the other hand, $\sigma \in \mathrm{st}_\Sigma(\tau)$, then $\mathrm{st}_{\widetilde{\Sigma}}(\tau) = \mathrm{stellar}_\sigma(\mathrm{st}_\Sigma(\tau))$, which is a star-shaped subdivision. Since $\mathrm{st}_\Sigma(\tau)$ is Lefschetz, so is $\mathrm{st}_{\widetilde{\Sigma}}(\tau)$, by Theorem 6.26.

Now suppose $\tau \notin \Sigma$, and let ρ denote the subdividing ray. Then $\rho \in \tau$, so $\mathrm{st}_{\widetilde{\Sigma}}(\tau) \subseteq \mathrm{st}_{\widetilde{\Sigma}}(\rho)$: in fact, $\mathrm{st}_{\widetilde{\Sigma}}(\tau) = \mathrm{st}_{\Sigma'}(\tau)$, where $\Sigma' = \mathrm{st}_{\widetilde{\Sigma}}(\rho)$. Since $\Sigma' = \mathrm{stellar}_\sigma(\mathrm{st}_\Sigma(\sigma))$, a star-shaped subdivision, Σ' is Lefschetz by Theorem 6.26, and it follows that $\mathrm{st}_{\widetilde{\Sigma}}(\tau)$ is Lefschetz too.

By Propositions 6.17 and 6.19, respectively, the fan $\widetilde{\Sigma}$ satisfies PD and HL. It remains to check that $\widetilde{\Sigma}$ satisfies HR as well.

Consider any $0 \leqslant k \leqslant d/2$ and $\ell \in \mathcal{K}(\Sigma)$. By Lemma 6.22, we have $\mathrm{hr}^k(\widetilde{\Sigma}, p^*\ell) = \mathrm{hr}^k(\Sigma, \ell) \oplus \mathrm{hr}^{k-1}(\mathrm{st}_\Sigma(\sigma), i_\sigma^*(\ell))$. The summands are nondegenerate, because Σ and $\mathrm{st}_\Sigma(\sigma)$ satisfy HL(ℓ) and HL($i_\sigma^*\ell$), respectively, so $\mathrm{hr}^k(\widetilde{\Sigma}, p^*\ell)$ is nondegenerate as well.

By the HR signature test (Theorem 6.13) the signature of $\mathrm{hr}^k(\widetilde{\Sigma}, p^*\ell)$ equals

$$\begin{aligned} &\sum_{i=0}^{k}(-1)^{k-i}\big(b_i(\Sigma) - b_{i-1}(\Sigma)\big) + \sum_{i=0}^{k-1}(-1)^{k-(i-1)}\big(b_{i-1}(\mathrm{st}_\Sigma(\sigma)) - b_{i-2}(\mathrm{st}_\Sigma(\sigma))\big) \\ &= \sum_{i=0}^{k}(-1)^{k-i}\big(b_i(\Sigma) + b_i(\mathrm{st}_\Sigma(\sigma)) - b_{i-1}(\Sigma) - b_{i-1}(\mathrm{st}_\Sigma(\sigma))\big) \\ &= \sum_{i=0}^{k}(-1)^{k-i}\big(b_i(\widetilde{\Sigma}) - b_{i-1}(\widetilde{\Sigma})\big). \end{aligned} \tag{6.7.1}$$

Lemma 6.11 states $p^*\ell \in \mathrm{cl}\,\mathcal{K}(\widetilde{\Sigma})$. Then there exists an open ball $U \subseteq A^1(\widetilde{\Sigma})$ containing $p^*\ell$ on which $\mathrm{hr}^k(\widetilde{\Sigma}, -)$ is nondegenerate. Choosing any $\ell' \in U \cap \mathcal{K}(\widetilde{\Sigma})$, we can use Corollary 6.14 to conclude that $\widetilde{\Sigma}$ satisfies HR^k. □

The converse is similar in spirit:

Proof of "$\Leftarrow$". Again, we argue by induction on dimension. The base case being trivial, we assume that, if $\widetilde{\Sigma}$ is Lefschetz and has dimension less than d, then Σ is Lefschetz as well. Now assume $\widetilde{\Sigma}$ is a Lefschetz fan of dimension d, and we show Σ is as well.

PD for Σ follows from Proposition 6.17. Next, consider a ray $\nu \in \Sigma(1)$. If $\nu \notin \overline{\mathrm{st}}_\Sigma(\sigma)(1)$, then $\mathrm{st}_\Sigma(\nu) = \mathrm{st}_{\widetilde{\Sigma}}(\nu)$, which is Lefschetz. If, on the other hand, $\nu \in \overline{\mathrm{st}}_\Sigma(\sigma)(1)$, then $\sigma \in \overline{\mathrm{st}}_\Sigma(\nu)(2)$, and $\overline{\mathrm{st}}_{\widetilde{\Sigma}}(\nu) = \mathrm{stellar}_\sigma(\overline{\mathrm{st}}_\Sigma(\nu))$. Since $\mathrm{st}_{\widetilde{\Sigma}}(\nu)$ is Lefschetz, so is $\mathrm{st}_\Sigma(\nu)$, by Theorem 6.26. Either way, $\mathrm{st}_\Sigma(\nu)$ has the HR property for each ray ν, so Σ has the HL property (by Proposition 6.19).

A similar argument shows that $\mathrm{st}_\Sigma(\tau)$ is Lefschetz for all cones τ of Σ: if the star remains a star in $\widetilde{\Sigma}$, it is Lefschetz by hypothesis. Otherwise, a subdivision of it is a star in $\widetilde{\Sigma}$. If $\tau = \sigma$, the subdivided edge, we invoke Theorem 6.26. Otherwise, we note the dimension is less than d, so $\mathrm{st}_\Sigma(\tau)$ is Lefschetz by induction.

It remains to establish HR^k for Σ, for $0 \leqslant k \leqslant d/2$. The condition is vacuous if $K(\Sigma) = \varnothing$. Otherwise, choose any $\ell \in K(\Sigma)$. By Lemma 6.22,

$$\mathrm{hr}^k(\widetilde{\Sigma}, p^*\ell) = \mathrm{hr}^k(\Sigma, \ell) \oplus \mathrm{hr}^{k-1}(\mathrm{st}_\Sigma(\sigma), i_\sigma^*(\ell)).$$

Since the second factor is the blowdown of $\mathrm{st}_{\widetilde{\Sigma}}(\rho)$, it is Lefschetz by Theorem 6.26, and the first factor is Lefschetz by the argument above. So both summands are nondegenerate, and so is $\mathrm{hr}^k(\widetilde{\Sigma}, p^*\ell)$.

By HR, the bilinear form $\mathrm{hr}^k(\widetilde{\Sigma}, \widetilde{\ell})$ has the expected signature for all $\widetilde{\ell} \in \mathcal{K}(\widetilde{\Sigma})$. It follows by Proposition 6.12 that $\mathrm{hr}^k(\widetilde{\Sigma}, p^*\ell)$ also has that signature, since it is nondegenerate and $p^*\ell$ lies in the boundary of $\mathcal{K}(\widetilde{\Sigma})$.

The HR property for $\mathrm{st}_\Sigma(\sigma)$ determines the signature of $\mathrm{hr}^{k-1}(\mathrm{st}_\Sigma(\sigma), i_\sigma^*(\ell))$, and we obtain the signature of $\mathrm{hr}^k(\Sigma, \ell)$ by subtraction. Using the calculation (6.7.1) again, we find that it equals $\sum_{i=0}^{k}(-1)^{k-i}\big(b_i(\Sigma) - b_{i-1}(\Sigma)\big)$, and we conclude Σ has the HR^k property. □

Putting the pieces together gives a proof that the Lefschetz property is an invariant of the support of a fan.

Theorem 1.6. Let Σ_1 and Σ_2 be simplicial fans that have the same support $|\Sigma_1| = |\Sigma_2|$. If $\mathcal{K}(\Sigma_1)$ and $\mathcal{K}(\Sigma_2)$ are nonempty, then Σ_1 is Lefschetz if and only if Σ_2 is Lefschetz.

Proof of Theorem 1.6. Suppose $|\Sigma| = |\Sigma'|$. According to Theorem 6.25, there is a sequence of fans $(\Sigma_0, \Sigma_1, \cdots, \Sigma_N)$ with $\Sigma = \Sigma_0$, $\Sigma_N = \Sigma'$, and for which either $\Sigma_i \to \Sigma_{i+1}$ or $\Sigma_{i+1} \to \Sigma_i$ is an edge subdivision, for each i. It is implicit in the argument of [Wło97] that edge subdivisions can be chosen in such a way that, whenever $\Sigma_i = \mathrm{stellar}_\sigma(\Sigma_{i+1})$, if $\mathcal{K}(\Sigma_i)$ is nonempty, then so is $\mathcal{K}(\Sigma_{i+1})$: see, for example, the discussion around [AKMW02, Theorem 0.3.1]. By Theorem 6.27, if any one of these fans is Lefschetz, then they all are. $\square$

In our terminology, the main result of [AHK18] says that the Bergman fan of M is Lefschetz. We use the result to show that the conormal fan of M is Lefschetz.

Lemma 6.28. If Σ_1 and Σ_2 are Lefschetz fans, then so is $\Sigma_1 \times \Sigma_2$.

Proof. It was shown in [AHK18, Section 7.2] that, if Σ_1 and Σ_2 have PD, HL, and HR, then so does $\Sigma_1 \times \Sigma_2$. Since stars of cones in a product are products of stars in the factors, we conclude that $\Sigma_1 \times \Sigma_2$ is a Lefschetz fan by induction on dimension. $\square$

Theorem 6.29. The conormal fan $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ of a loopless and coloopless matroid M is Lefschetz.

Proof. Since the Bergman fan is Lefschetz, from Lemma 6.28 we see the fan $\Sigma_{\mathrm{M}} \times \Sigma_{\mathrm{M}^\perp}$ is Lefschetz. Moreover, its support is equal to that of $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$. Bergman fans are quasiprojective, since they are subfans of the permutohedral fan, so $\mathcal{K}(\Sigma_{\mathrm{M}} \times \Sigma_{\mathrm{M}^\perp})$ is nonempty. We saw that the bipermutohedral fan $\Sigma_{E,\overline{E}}$ is the normal fan of the bipermutohedron, so the conormal fan is also quasiprojective, and $\mathcal{K}(\Sigma_{\mathrm{M},\mathrm{M}^\perp})$ is nonempty as well. By Theorem 1.6, then, $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ is Lefschetz. □

The extra structure present in the Chow rings of Lefschetz fans leads easily to an Aleksandrov–Fenchel-type inequality.

Theorem 6.30. Let Σ be a Lefschetz fan of dimension d, and $\ell_2, \ell_3, \dots, \ell_d$ elements of $\operatorname{cl}\mathcal{K}(\Sigma)$. Then for any $\ell_1 \in A^1(\Sigma)$,

$$\deg(\ell_1\ell_2\cdots\ell_d)^2 \geqslant \deg(\ell_1\ell_1\ell_3\cdots\ell_d)\cdot\deg(\ell_2\ell_2\ell_3\cdots\ell_d). \tag{6.7.2}$$

Proof. We first verify the inequality when $\ell_i \in \mathcal{K}(\Sigma)$ for each $2 \leqslant i \leqslant d$. For this, let $L = \ell_3\cdots\ell_d$, a Lefschetz element, and consider $\langle -,-\rangle \coloneqq \langle -,-\rangle_L$ on $A^1(\Sigma)$.

If $\langle \ell_2,\ell_2\rangle \neq 0$, let $\ell_1' = \ell_1 - \frac{\langle \ell_1,\ell_2\rangle}{\langle \ell_2,\ell_2\rangle}\ell_2$, so that $\langle \ell_1',\ell_2\rangle = 0$. This means $\ell_1' \in PA^1(\Sigma,\ell_2)$, so by HR,

$$\begin{aligned} 0 &\leqslant \left\langle \ell_1',\ell_1'\right\rangle \\ &= \left\langle \ell_1,\ell_1'\right\rangle \\ &= \langle \ell_1,\ell_1\rangle - \frac{\langle \ell_1,\ell_2\rangle}{\langle \ell_2,\ell_2\rangle}\langle \ell_1,\ell_2\rangle . \end{aligned}$$

By the signature test, $\langle -,-\rangle$ is negative-definite on the orthogonal complement of ℓ_1'. Therefore $\langle \ell_2,\ell_2\rangle < 0$, and we see

$$\langle \ell_1,\ell_2\rangle^2 \geqslant \langle \ell_1,\ell_1\rangle\cdot\langle \ell_2,\ell_2\rangle ,$$

which is equivalent to (6.7.2). (If, on the other hand, $\langle \ell_2,\ell_2\rangle = 0$, this inequality is obvious.)

Now we relax the hypothesis to consider $\ell_2,\dots,\ell_d \in \operatorname{cl}\mathcal{K}(\Sigma)$. The inequality (6.7.2) continues to hold by continuity, as in [AHK18, Theorem 8.8]. □

Theorem 1.4. For any matroid M, the h-vector of the broken circuit complex of M is log-concave.

Proof. It suffices to assume that M is loopless and coloopless. The classes $\gamma = \gamma_i$ and $\delta = \delta_i$ are pullbacks of the nef classes $\alpha = \alpha_i \in A^1(\Sigma_{\mathrm{M}})$ and $\alpha = \alpha_i \in A^1(\Delta_E)$, along the two maps $\pi : \Sigma_{\mathrm{M},\mathrm{M}^\perp} \to \Sigma_{\mathrm{M}}$ and $\mu : \Sigma_{\mathrm{M},\mathrm{M}^\perp} \to \Delta_E$, respectively. The pullback of a convex function on a fan is convex, so both γ and δ represent nef classes on the conormal fan. Since $\mathcal{K}(\Sigma_{\mathrm{M},\mathrm{M}^\perp})$ is nonempty, we see that $\gamma,\delta \in \operatorname{cl}\mathcal{K}(\Sigma_{\mathrm{M},\mathrm{M}^\perp})$, following the discussion at the end of Section 6.2.

By Theorem 1.2, we have

$$h_{r-k}(\mathrm{BC}(\mathrm{M})) = \deg_{\Sigma_{\mathrm{M},\mathrm{M}^\perp}}(\gamma^k \delta^{n-k-1})$$
$$= \langle \gamma, \delta \rangle_L \,,$$

where $L = \gamma^{k-1}\delta^{n-k-2}$. Since $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ is Lefschetz (Theorem 6.29) the log-concave inequalities follow from Theorem 6.30. □

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