Algebraic structures on polytopes

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My talk at the 2018 Joint Math Meetings in San Diego will discuss the algebraic and combinatorial structure of a beautiful family of polytopes. It is based on joint work with Marcelo Aguiar. It will be accessible to undergraduates and will not assume any previous knowledge of these topics.

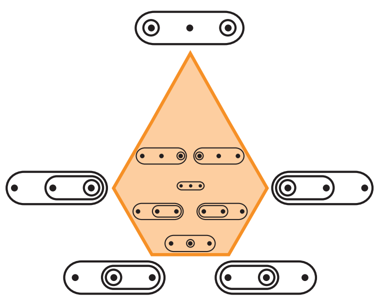

Figure 1: The two-dimensional associahedron and the labeling of its faces by tubings.

Two classical problems: inverting power series.

1. Multiplicative Inversion. Consider power series $A(x) = 1 + \sum_{n\geq 1} a_n \frac{x^n}{n!}$ and $B(x) = 1 + \sum_{n\geq 1} b_n \frac{x^n}{n!}$ that are multiplicative inverses; *i.e.*, $A(x)B(x) = 1$. The first few coefficients of $B(x) = 1/A(x)$ are:

$$\begin{aligned} b_1 &= -a_1 \\ b_2 &= -a_2 + 2a_1^2 \\ b_3 &= -a_3 + 6a_2a_1 - 6a_1^3 \\ b_4 &= -a_4 + 8a_3a_1 + 6a_2^2 - 36a_2a_1^2 + 24a_1^4 \end{aligned}$$

2. Compositional Inversion. Consider power series $C(x) = x + \sum_{n\geq 2} c_{n-1}x^n$, $D(x) = x + \sum_{n\geq 2} d_{n-1}x^n$ that are compositional inverses; that is, $C(D(x)) = x$. The first few coefficients of $D(x) = C(x)^{\langle -1\rangle}$ are:

$$\begin{aligned} d_1 &= -c_1 \\ d_2 &= -c_2 + 2c_1^2 \\ d_3 &= -c_3 + 5c_2c_1 - 5c_1^3 \\ d_4 &= -c_4 + 6c_3c_1 + 3c_2^2 - 21c_2c_1^2 + 14c_1^4 \end{aligned}$$

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Supported by NSF CAREER grant DMS-0956178 and Combinatorics grants DMS-0801075, DMS-1440140, and DMS-1600609.

It is natural to ask: What do these coefficients count? Two families of polytopes hold the answers.

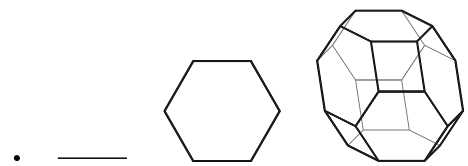

Figure 2: The permutahedra $\pi_1, \pi_2, \pi_3, \pi_4, \ldots$, whose face structures tell us how to compute the multiplicative inverse of a series.

The *permutahedron* π_n (see Figure 2) is the polytope in $\mathbb{R}^n$ whose vertices are the $n!$ permutations of $\{1, 2, \ldots, n\}$, regarded as vectors. Every face of a permutahedron is a product of permutahedra of lower dimensions.

The face structure of permutahedra tells us how to compute $B(x) = 1/A(x)$. For example, the formula for b_4 shown above is determined by the faces of π_4:

- 1 truncated octahedron π_4,
- 8 hexagons $\pi_3 \times \pi_1$ and 6 squares $\pi_2 \times \pi_2$,
- 36 segments $\pi_2 \times \pi_1 \times \pi_1$, and
- 24 points $\pi_1 \times \pi_1 \times \pi_1 \times \pi_1$.

The signs are given by the dimensions of the faces.

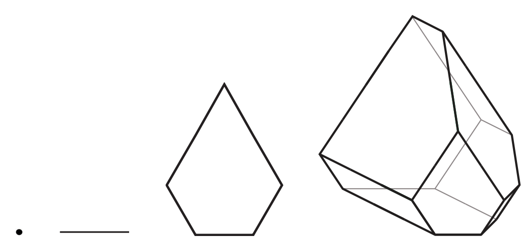

Figure 3: The associahedra $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4, \ldots$, whose face structures tell us how to compute the compositional inverse of a series.

The *associahedron* $\mathfrak{a}_n$ (see Figure 3) is a polytope in $\mathbb{R}^n$ whose vertices correspond to the $C_n = \frac{1}{n+1}\binom{2n}{n}$ associations of a product $x_1 \cdots x_n$ into binary products. For example, when $n = 5$, one such association is $(x_1x_2)((x_3x_4)x_5)$. Every face of an associahedron is a product of associahedra of smaller dimensions.

The face structure of permutahedra tells us how to compute $D(x) = C(x)^{\langle -1 \rangle}$. For instance, the formula for d_4 shown above comes from the faces of $\mathfrak{a}_4$:

- 1 three-dimensional associahedron $\mathfrak{a}_4$,
- 6 pentagons $\mathfrak{a}_3 \times \mathfrak{a}_1$ and 3 squares $\mathfrak{a}_2 \times \mathfrak{a}_2$,
- 21 segments $\mathfrak{a}_2 \times \mathfrak{a}_1 \times \mathfrak{a}_1$, and
- 14 points $\mathfrak{a}_1 \times \mathfrak{a}_1 \times \mathfrak{a}_1 \times \mathfrak{a}_1$.

Again, the signs come from the face dimensions.

We discovered these results unexpectedly, when studying a more general family of polytopes.

Hopf monoids and generalized permutahedra.

Edmonds, Stanley, and others taught us that to study combinatorial objects, it is often helpful to build polyhedral models for them. *Generalized permutahedra* (or equivalently, submodular functions) are a particularly useful family of polyhedra. They are deformations of the permutahedron, obtained by moving the faces while preserving their directions. Two important families are the permutahedra and associahedra, but there are many more. Figure 4 shows a few three dimensional examples.

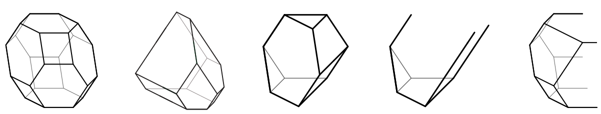

Figure 4: Five generalized permutahedra.

Joni and Rota, Joyal, Stanley, and others showed that to study combinatorial objects, it is often helpful to endow them with algebraic structures. In their book "Monoidal functors, species and Hopf algebras" (2010), Aguiar and Mahajan provide a particularly useful framework: *Hopf monoids in species.* This framework applies to families of combinatorial objects having natural operations of *merging* two disjoint objects into one, and *breaking* an object into two disjoint parts, subject to various axioms.

Many families of combinatorial objects carry such polyhedral and algebraic structures: graphs, matroids, posets, set partitions, and simplicial complexes, to name just a few. The main idea of this project is to bring together these two points of view:

Theorem 1. *Generalized permutahedra form a Hopf monoid in species* GP*. In fact, they are the universal family of polyhedra with this algebraic structure. The Hopf monoid* GP *contains or projects to the Hopf monoids* $\mathrm{G}, \mathrm{M}, \mathrm{P}, \Pi, \mathrm{SC}$ *of graphs, matroids, posets, set partitions, and simplicial complexes.*

Combinatorial Hopf monoids and their antipodes.

A key component of a Hopf monoid is its *antipode map* S, which is analogous to the inverse map in a group. The antipode is given by a very large alternating sum, generally involving lots of cancellation. A fundamental and often difficult question is to compute this antipode. Figure 5 shows the antipode of a graph and of a poset.

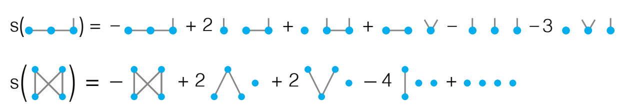

Figure 5: The antipode of a graph and of a poset.

These formulas result from simplifying alternating sums of 13, and 75 terms, respectively. How can we systematically explain the extensive cancellation that is taking place?

This is a rather subtle question. For instance, although most of the Hopf algebraic structures $\mathrm{G}, \mathrm{M}, \mathrm{P}, \Pi, \mathrm{SC}$ have been known for decades, optimal formulas for their antipodes were not known until very recently.

Fortunately, generalized permutahedra provide a geometric setting where the cancellation may be completely understood topologically. This allows us to obtain the optimal formula for the antipode of generalized permutahedra.

Theorem 2. *The antipode of the Hopf monoid* GP *is given by the following* ***cancellation-free*** *and* ***grouping-free*** *formula: If $\mathfrak{p}$ is a generalized permutahedron in $\mathbb{R}^n$ then*

$$\mathrm{S}(\mathfrak{p}) = \sum_{\mathfrak{q} \text{ face of } \mathfrak{p}} (-1)^{n - \dim \mathfrak{q}} \, \mathfrak{q}.$$

This implies optimal formulas for the antipodes of $\mathrm{G}, \mathrm{M}, \mathrm{P}, \Pi,$ *and* SC.

Inverting formal power series, revisited.

Once we discovered these results, we asked: What happens when we apply this general theory to the polytopes $\pi_1, \pi_2, \pi_3, \pi_4, \ldots$ and $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4, \ldots$? The answer was very surprising to us: Theorem 2 implies that the multiplicative and compositional inverses of power series are given by the alternating sum of the faces of permutahedra and associahedra.

Figure 6: Federico Ardila works in algebraic, geometric, and topological combinatorics, and on helping build diverse, compassionate, and equitable mathematical communities. He has advised 40 thesis students in the US and Colombia, including 15 women and 30 members of underrepresented groups. When he is not at work he is probably on a fútbol field, or treasure hunting in little record stores and sharing what he's found with La Pelanga DJ Collective. (Photo courtesy of May-Li Khoe.)